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«БРЕСТСКИЙ ГОСУДАРСТВЕННЫЙ ТЕХНИЧЕСКИЙ УНИВЕРСИТЕТ»

КАФЕДРА ВЫСШЕЙ МАТЕМАТИКИ

Precalculus Mathematics for Calculus (part 1)

учебно-методическая разработка на английском языке
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Настоящая методическая разработка предназначена для иностранных слушателей факультета довузовской подготовки. Данная разработка содержит необходимый материал по темам «Числовые множества», «Арифметические вычисления», «Алгебраические преобразования», «Рациональные уравнения», «Системы рациональных уравнений». Изложение теоретического материала сопровождается рассмотрением большого количества примеров и задач, некоторые понятия и примеры проиллюстрированы.

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Section1. The Real Number System

1.1. SETS

Human beings share the desire to organize and classify. Ancient astronomers classified stars into groups called *constellations*. Modern astronomers continue to classify stars by such characteristics as color, mass, size, temperature, and distance from Earth. In mathematics it is useful to place numbers with similar characteristics into ***sets***. The following sets of numbers are used extensively in the study of algebra.

Natural numbers (N)	$\{1, 2, 3, 4, ...\}$
Integers (Z)	$\{..., -4, -3, -2, -1, 0, 1, 2, 3, 4, ...\}$
Rational numbers (Q)	{all terminating or repeating decimals}
Irrational numbers (I)	{all nonterminating, nonrepeating decimals}
Real numbers (R)	{all rational or irrational numbers}

If a number in decimal form terminates or repeats a block of digits, then the number is a rational number. Here are two examples of rational numbers.

$0,75$ is a terminating decimal.

$0,2\overline{45}$ is a repeating decimal. The bar over the 45 means that the block of digits 45 repeats without end. That is, $0,2\overline{45} = 0,2454545...$.

Rational numbers can be written in the form $\frac{p}{q}$ where p and q are integers and $q \neq 0$. The examples of rational numbers written in this form are

$$\frac{3}{4}\ ;\ \frac{27}{115}\ ;\ -\frac{5}{2}\ ;\ \frac{7}{1}$$

Note that $\frac{7}{1} = 7$ and, in general, $\frac{n}{1} = n$ for any integer n. Therefore, all integers are rational numbers.

When a rational number is written in the form of $\frac{p}{q}$ the decimal form of the rational number can be found by dividing the numerator by the denominator.

$$\frac{3}{4} = 0,75\ ;\ \frac{27}{110} = 0,2\overline{45}$$

In its decimal form, an irrational number neither terminates nor repeats. For example, $0,27277277727777...$ is a nonterminating, nonrepeating decimal and thus is an irrational number. One of the best-known irrational numbers is pi, denoted by the Greek symbol π. The number π is defined as the ratio of the circumference of a circle to its diameter. The rational number 3,14 or the rational number $\frac{22}{7}$ is often used as an approximation of the irrational number π.

Every real number is either a rational number or an irrational number. If a real number is written in its decimal form, it is a terminating decimal, a repeating decimal, or a nonterminating and nonrepeating decimal.

The relationships among various sets of numbers are shown in Figure 1.

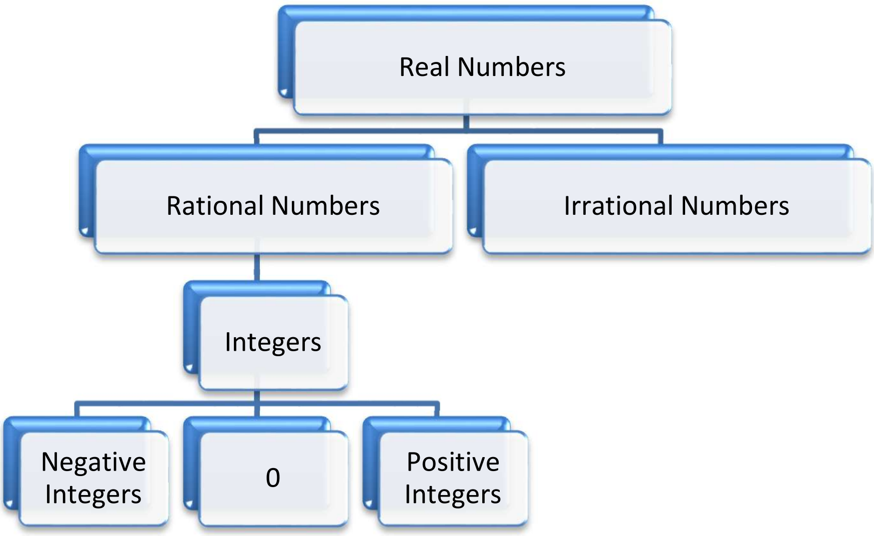

Figure 1

EXAMPLE 1 Write each rational number in the form $\frac{a}{b}$ where a and b are integers, and $b \neq 0$.

(a) 0,4 (b) 7,75 (c) 0,0007 (d) 530,0276

SOLUTION

(a) $0,4 = \frac{4}{10} = \frac{4:2}{10:2} = \frac{2}{5}$ (b) $7,75 = 7\frac{75}{100} = 7\frac{75:25}{100:25} = 7\frac{3}{4}$

(c) $0,0007 = \frac{7}{10000}$ (d) $530,0276 = 530\frac{276}{10000} = 530\frac{276:4}{10000:4} = 530\frac{69}{2500}$

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EXAMPLE 2 Write each rational number as a terminating decimal.

(a) $\frac{27}{100}$; (b) $9\frac{7}{100}$; (c) $\frac{277}{10}$; (d) $\frac{2}{5}$; (e) $\frac{7}{25}$; (f) $14\frac{7}{250}$

SOLUTION

(a) $\frac{27}{100} = 27:100 = 0,27$; (b) $9\frac{7}{100} = 9 + 7:100 = 9 + 0,07 = 9,07$

(c) $\frac{277}{10} = 277:10 = 27,7$; (d) $\frac{2}{5} = 2:5 = 0,4$; (e) $\frac{7}{25} = 7:25 = 0,28$

(f) $14\frac{7}{250} = 14 + \frac{7}{250} = 14 + 7:250 = 14 + 0,028 = 14,028$

Prime numbers and *composite numbers* play an important role in almost every branch of mathematics.

Definition A **prime number** is a positive integer greater than 1 that has no positive integer factors other then itself and 1.

The ten smallest prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23 and 29. Each of these numbers has only itself and 1 as factors.

Definition A **composite number** is a positive integer greater than 1 that is not a prime number.

For example, 10 is a composite number because 10 has both 2 and 5 as factors. The ten smallest composite numbers are 4, 6, 8, 9, 10, 12, 14, 15, 16 and 18.

EXAMPLE 3 **(Classify Real Numbers)**. Determine which of the following numbers are:

(a) integers; (b) rational numbers; (c) irrational numbers; (d) real numbers; (e) prime numbers; (f) composite numbers.

SOLUTION

a. Integers: 0; 6; 7; 41; 51.

b. Rational numbers: $-0.2; 0; 0.3; 0.\overline{4}; 6; 51; 47$.

c. Irrational numbers: $7.098765112...; \pi$

d. Real numbers: $-0.2; 51; 0.2\overline{45}; \pi; 7.098765112...$

e. Prime numbers: 7, 41.

f. Composite numbers: 6, 51.

Each member of a set is called an **element** of the set. For instance, if $C = \{2,3,5\}$ then the elements of C are 2, 3 and 5. The notation $2 \in C$ is read as "2 is an element of C".

Set A is a subset of set B if every element of A is also an element of B, and we write $A \subseteq B$. For instance, the set of ***negative integers*** $\{-1,-2,-3,-4,...\}$ is a subset of the set of integers. The set of ***positive integers*** $\{1,2,3,4,...\}$ (the natural numbers) is also a subset of the set of integers.

The ***empty set***, or the ***null set***, is the set that contains no elements. The symbol $\varnothing$ is used to represent the empty set. The set of people who have run a 2-minute mile is the empty set.

The set of natural numbers that are less than 6 is $\{1,2,3,4,5\}$. This is the example of a ***finite set***; all the elements of the set can be listed. The set of all natural numbers is an example of an ***infinite set***. There is no largest natural number, so all the elements of the set of natural numbers cannot be listed.

Sets are often written using ***set-builder*** notation. Set-builder notation can be used to describe almost any set, but it is especially useful when writing infinite sets. For instance, the set

$$\{2n \mid n \in N\}$$

is read as "the set of elements $2n$ such that n is a natural number". By replacing n with each of the natural numbers, it becomes the set of positive even integers: $\{2,4,6,8,...\}$.
The set of real numbers that are greater than 2 is written

$$\{x \mid x > 2, x \in R\}$$

and is read "the set of x such that x is greater than 2 and x is an element of the real numbers".

Much of the work we do in this text uses the real numbers. Taking with into account, we will frequently write, for instance, $\{x \mid x > 2, x \in R\}$ in a shortened form as $\{x \mid x > 2\}$, where we assume that x is a real number.

EXAMPLE 4 (Use Set-Builder Notation). List four smallest elements in $\{n^3 \mid n \in N\}$.

SOLUTION

Because we want four *smallest* elements, we choose four smallest natural numbers. Thus $n = 1,2,3$ and 4. Therefore, four smallest elements of the set $\{n^3 \mid n \in N\}$ are 1, 8, 27 and 64.

1.2. UNION AND INTERSECTION OF SETS

Just as operations such as addition and multiplication are performed on real numbers, operations are performed on sets. Two operations performed on sets are union and intersection. The union of two sets A and B is the set of elements that belong to A or to B, or to both A and B.

Definition (Union of Two Sets) The union of two sets, written $A \cup B$ is the set of all elements that belong to either A or B. In a set-builder notation, this is written as

$$A \cup B = \{x \mid x \in A, \text{ or } x \in B\}.$$

EXAMPLE 5 For the given $A = \{2,3,4,5\}$ and $B = \{0,1,2,3,4\}$ find $A \cup B$.

SOLUTION

$$A \cup B = \{0,1,2,3,4,5\}.$$

Note that an element that belongs to both sets is listed only once.

The intersection of two sets A and B is the set of elements that belong to both A and B.

Definition (Intersection of Two Sets) The intersection of two sets, written $A \cap B$ is the set of all elements that are common to both A and B. In a set-builder notation, this is written as

$$A \cap B = \{x \mid x \in A, \text{ and } x \in B\}.$$

EXAMPLE 6 For the given $A = \{2,3,4,5\}$ and $B = \{0,1,2,3,4\}$ find $A \cap B$.

SOLUTION

$$A \cap B = \{2,3,4\}.$$

The intersection of two sets contains the elements common to both sets.

If the intersection of two sets is the empty set, the two sets are said to be ***disjoint***. For example, if $A=\{2,3,4,5\}$ and $B=\{6,7,8\}$ then $A\cap B=\varnothing$ and A and B are disjoint sets.

1.3. OPERATIONS OF ARITHMETIC

• **Addition** of two real numbers a and b is designated by $a+b$. If $a+b=c$, then c is the ***sum*** and real numbers a and b are called ***terms***.

• **Multiplication** of two real numbers a and b is designated by $a\cdot b$. If $a\cdot b=c$, then c is the ***product*** and real numbers a and b are called ***factors.***

• The number $-b$ is referred to as the ***additive inverse*** of b. The **subtraction** designated by $a-b$ can be performed by adding a and the additive inverse of b. That is, $a-b=a+(-b)$. If $a-b=c$, then c is called the ***difference*** of a and b.

• Two real numbers whose product is 1 are called ***multiplicative inverse*** or ***reciprocals*** of each other. The reciprocal of the nonzero number b is $\frac{1}{b}$. The **division** of a and b designated by $a:b$ with $b\neq 0$ can be performed by multiplying a and the reciprocal of b. That is $a:b=\frac{a}{b}=a\cdot\frac{1}{b}$. If $a:b=c$ then c is called the **quotient** of a and b

The notation $a:b$ is often represented by the fractional notation $\frac{a}{b}$. The real number a is the **numerator**, and the nonzero real number b is the **denominator** of the fraction.

Properties of Real Numbers

Let a,b and c be real numbers.

	Addition Properties	***Multiplication Properties***
Closure	$a+b$ is a unique real number	$a\cdot b$ is a unique real number
Commutative	$a+b=b+a$	$a\cdot b=b\cdot a$
Associative	$(a+b)+c=a+(b+c)$	$(ab)c=a(bc)$
Identity	There exists a unique real number 0 such that $a+0=0+a=a$	There exists a unique real number 1 such that $a\cdot 1=1\cdot a=a$
Inverse	For each real number a there is an unique real number $-a$ such that $a+(-a)=0$	For each nonzero real number a there is an unique real number $\frac{1}{a}$ such that $a\cdot\frac{1}{a}=1$
Distributive	$a\cdot(b+c)=ab+ac$	$a\cdot(b+c)=ab+ac$

Exponential Expressions

A compact method of writing $2\cdot2\cdot2\cdot2\cdot2$ is 2^5. The expression 2^5 is written in ***exponential notation***. Similarly, we can write

$$\frac{5x}{2}\cdot\frac{5x}{2}\cdot\frac{5x}{2} \quad \text{as} \quad \left(\frac{5x}{2}\right)^3.$$

Exponential notation can be used to express the product of any expression that is used repeatedly as a factor.

> ***Definition of Natural Number Exponents*** If b is any real number and n is a natural number, then $b^n = \underbrace{b\cdot b\cdot b\cdot\ldots\cdot b}_{n-times}$, where b is the ***base*** and n is the ***exponent***.

EXAMPLE 7 Evaluate (a) $(-3^4)(-4)^2$; (b) $\dfrac{-4^3}{(-4)^3}$

SOLUTION

(a) $(-3^4)(-4)^2 = (-3\cdot3\cdot3\cdot3)\cdot(-4)\cdot(-4) = -81\cdot16 = -1296$.

(b) $\dfrac{-4^3}{(-4)^3} = \dfrac{-4\cdot4\cdot4}{(-4)\cdot(-4)\cdot(-4)} = \dfrac{-64}{-64} = 1$.

The Order of Operations Agreement

If grouping symbols are present, first perform the operations within the grouping symbols, innermost grouping symbol first, while observing the order given in steps 1 to 3.

Step 1 Evaluate exponential expressions.

Step 2 Do multiplication and division as they occur from left to right.

Step 3 Do addition and subtraction as they occur from left to right.

One of the ways in which the Order of Operations Agreement is used is to evaluate variable expressions. The addends of a variable expression are called **terms**. The terms for the expression $3x^2-4xy+2x-5y-7$ are $3x^2$, $-4xy$, $2x$, $-5y$ and -7. Observe that the sign of a term is the sign that immediately precedes it.

The terms $3x^2$, $-4xy$, $2x$ and $-5y$ are **variable terms**. The term -7 is a constant term. Each variable term has a numerical coefficient and a variable part. The numerical coefficient for the term $3x^2$ is 3; the numerical coefficient for the term $-4xy$ is -4, the numerical coefficient for the term $2x$ is 2; and the numerical coefficient for the term $-5y$ is -5. When the numerical coefficient is 1 or -1 (as in x and $-x$), the 1 is usually not written.

To **evaluate** a variable expression, replace the variables by their given values and then use the Order of Operations Agreement to simplify the result.

We can identify which properties of real numbers have been used to rewrite an expression by closely comparing the original and final expressions and finding any changes. For instance, to simplify $(6x)2$ both the commutative property and associative property of multiplication are used.

$$(6x)\cdot 2 = 2\cdot(6x) = (2\cdot 6)x = 12x$$

Terms that have the same variable part are called **like terms**. The distributive property is also used to simplify an expression with like terms such as $3x^2 + 9x^2 = (3+9)x^2 = 12x^2$.

Note from this example that *like terms are combined by adding the coefficients of the like terms.*

Expressions with Grouping Symbols

In mathematics, ***parentheses ( ) act as grouping symbols, giving different meanings*** to expressions. For example, $(4\cdot 6)+7$ means "add 7 to the product of 4 and 6 ", while $4\cdot(6+7)$ means "multiply the sum of 6 and 7 by 4". When simplifying any numerical expression, always perform the operations within parentheses first.

$$(4\cdot 6)+7 = 24+7 = 31 \qquad 4\cdot(6+7) = 4\cdot 13 = 52$$

Besides parentheses, other symbols are used to indicate grouping, such as brackets [ ]**. *The expressions 2(5+9) and 2[5+9] have the same meaning*:** 2 is multiplied by the sum of 5 and 9. A *bar, or fraction line, also acts as a symbol of* grouping, telling us to perform the operations in the numerator and/or denominator first.

$$\frac{20-8}{3} = \frac{12}{3} = 4 \qquad \frac{6}{3+1} = \frac{6}{4} = 1\frac{1}{2}.$$

EXAMPLE 8 Evaluate $4\cdot 11^2 - 3\cdot(-3^2 - 2^4):15$.

SOLUTION

$$4\cdot 11^2 - 3\cdot(-3^2-2^4):15 = 4\cdot 11^2 - 3\cdot(-3\cdot 3 - 2\cdot 2\cdot 2\cdot 2):15 = 4\cdot 11^2 - 3\cdot(-9-16):15 =$$

$$= 4\cdot 11^2 + 3\cdot 25:15 = 4\cdot 11\cdot 11 + 75:15 = 44\cdot 11 + 5 = 484 + 5 = 489$$

EXAMPLE 9 (Evaluate a Variable Expression)

Evaluate: (a) $\dfrac{x^3 - y^3}{x^2 + xy + y^2}$ when $x = 3$, $y = 2$;

(b) $(x-2y)^2 - 3z$ when $x = 3$, $y = 2$, $z = 4$.

SOLUTION

(a) $\dfrac{x^3 - y^3}{x^2 + xy + y^2}$: $\quad \dfrac{3^3 - 2^3}{3^2 + 3\cdot 2 + 2^2} = \dfrac{27-8}{9+6+4} = \dfrac{19}{19} = 1$

(b) $(x-2y)^2-3z$: $(3-2\cdot 2)^2-3\cdot 4=(-1)^2-12=1-12=-11$.

Properties of Fractions

For all fractions $\dfrac{a}{b}$ and $\dfrac{c}{d}$, where $b\neq 0$ and $d\neq 0$:

Equality	$\dfrac{a}{b}=\dfrac{c}{d}$ *if and only if* $ad=cb$
Equivalent fractions	$\dfrac{a}{b}=\dfrac{ac}{bc}$, $c\neq 0$
Addition	$\dfrac{a}{b}+\dfrac{c}{b}=\dfrac{a+c}{b}$
Substraction	$\dfrac{a}{b}-\dfrac{c}{b}=\dfrac{a-c}{b}$
Multiplication	$\dfrac{a}{b}\cdot\dfrac{c}{d}=\dfrac{ac}{bd}$
Division	$\dfrac{a}{b}:\dfrac{c}{d}=\dfrac{a}{b}\cdot\dfrac{d}{c}=\dfrac{ad}{bc}$
Sign properties	$\dfrac{-a}{b}=\dfrac{a}{-b}=-\dfrac{a}{b}$

Division properties of Zero:

- zero divided by any nonzero number b is zero $\dfrac{0}{b}=0$;

- division by zero is undefined.

EXAMPLE 10 Use the properties of fractions or division properties of zero to find the following indicated sums, differences, products, or quotients. Assume that a is a nonzero real number.

(a) $\dfrac{2a}{3}-\dfrac{a}{5}$; (b) $\dfrac{2a}{5}\cdot\dfrac{3a}{4}$; (c) $\dfrac{5a}{6}:\dfrac{3a}{4}$; (d) $\dfrac{0}{3a}$

SOLUTION

(a) Because fractions do not have a common denominator, we first rewrite each fraction as equivalent fraction with common denominator 15. This is accomplished by multiplying both the numerator and the denominator of $\dfrac{2a}{3}$ by 5 and by multiplying both the numerator and the denominator of $\dfrac{a}{5}$ by 3.

$$\frac{2a}{3}-\frac{a}{5}=\frac{2a\cdot 5}{3\cdot 5}-\frac{a\cdot 3}{5\cdot 3}=\frac{10a}{15}-\frac{3a}{15}=\frac{10a-3a}{15}=\frac{7a}{15}$$

(b) $\quad \dfrac{2a}{5}\cdot\dfrac{3a}{4}=\dfrac{2a\cdot 3a}{5\cdot 4}=\dfrac{6a^2}{20}=\dfrac{3a^2}{10}$

(c) $\quad \dfrac{5a}{6}:\dfrac{3a}{4}=\dfrac{5a}{6}\cdot\dfrac{4}{3a}=\dfrac{5a\cdot 4}{6\cdot 3a}=\dfrac{20a}{18a}=\dfrac{10}{9}$

(d) $\quad \dfrac{0}{3a}=0.$

EXAMPLE 11 Evaluate

(a) $\dfrac{6}{23}+\dfrac{7}{23}+\dfrac{10}{23}$; (b) $\dfrac{10}{51}+\dfrac{5}{9}$; (c) $\dfrac{5}{24}+\dfrac{7}{60}$; (d) $7\dfrac{5}{6}+3\dfrac{3}{4}$

(e) $6\dfrac{7}{9}-5\dfrac{1}{6}$; (f) $62-3\dfrac{7}{15}$.

SOLUTION

(a) $\quad \dfrac{6}{23}+\dfrac{7}{23}+\dfrac{10}{23}=\dfrac{6+7+10}{23}=\dfrac{23}{23}=23:23=1$

(b) $\quad \dfrac{10}{51}+\dfrac{5}{9}=\dfrac{10\cdot 3}{153}+\dfrac{5\cdot 17}{153}=\dfrac{30}{153}+\dfrac{85}{153}=\dfrac{115}{153}$

(c) $\quad \dfrac{5}{24}+\dfrac{7}{60}=\dfrac{5\cdot 5}{120}+\dfrac{7\cdot 2}{120}=\dfrac{25}{120}+\dfrac{14}{120}=\dfrac{39}{120}=\dfrac{39:3}{120:3}=\dfrac{13}{40}$

(d) $\quad 7\dfrac{5}{6}+3\dfrac{3}{4}=7+\dfrac{5}{6}+3+\dfrac{3}{4}=7+\dfrac{10}{12}+3+\dfrac{9}{12}=(7+3)+\left(\dfrac{10}{12}+\dfrac{9}{12}\right)=$

$=10\dfrac{19}{12}=11\dfrac{7}{12}$

(e) $\quad 6\dfrac{7}{9}-3\dfrac{1}{6}=\left(6+\dfrac{14}{18}\right)-\left(3+\dfrac{3}{18}\right)=6+\dfrac{14}{18}-3-\dfrac{3}{18}=(6-3)+\left(\dfrac{14}{18}-\dfrac{3}{18}\right)=3+\dfrac{11}{18}=3\dfrac{11}{18}$

(f) $\quad 62-3\dfrac{7}{15}=62-\left(3+\dfrac{7}{15}\right)=62-3-\dfrac{7}{15}=59-\dfrac{7}{15}=58+1-\dfrac{7}{15}=58+\dfrac{15}{15}-\dfrac{7}{15}=$

$=58+\dfrac{15-7}{15}=58+\dfrac{8}{15}=58\dfrac{8}{15}$

Exercise Set 1.1

In Exercises 1,2 and 3, determine which of the numbers are **a.** integers, **b.** rational numbers, **c.** irrational numbers, **d.** real numbers.

1. $\quad 0 \quad 4 \quad \dfrac{1}{5} \quad \dfrac{11}{3} \quad \sqrt{4} \quad \sqrt{9} \quad 3{,}1\overline{4} \quad 3{,}14 \quad -0{,}272272227\ldots$

2. $\quad 2{,}8\overline{10} \quad -4{,}25 \quad -\dfrac{1}{4} \quad \dfrac{10}{2} \quad \dfrac{2}{10} \quad \pi \quad 0{,}131313\ldots \quad 0{,}131131113\ldots \quad \dfrac{0}{4}$

3.

0.36	$0.36363636\ldots$	$0.3\overline{6}$	$0.363363336\ldots$	$\sqrt{48}$
$\sqrt{8}$	10π	$0{,}121314\ldots$	$\sqrt{16}$	$\sqrt{49}$
$0{,}989989998\ldots$	$0{,}725$	$\sqrt{121}$	$\pi+30$	$0{,}246\overline{82}$
$-5{,}28$	$0{,}141414\ldots$	$-\sqrt{5}$	$-\pi$	$\pi-2$

In 4–13, write each rational number in the form $\frac{a}{b}$ where a and b are integers, and $b \neq 0$.

4. 0,7 **5.** 0,18 **6.** –0,21 **7.** 9 **8.** –3

9. 0 **10.** $5\frac{1}{2}$ **11.** $-3\frac{1}{3}$ **12.** 0,007 **13.** –2,3

In 14–23, write each rational number as a repeating decimal. (Hint: Every terminating decimal has a repeating zero, for example, $0.3 = 0.3\overline{0}$.)

14. $\frac{5}{8}$ **15.** $\frac{9}{4}$ **16.** $-5\frac{1}{2}$ **17.** $\frac{13}{8}$ **18.** $-\frac{7}{12}$

19. $\frac{5}{3}$ **20.** $\frac{7}{9}$ **21.** $\frac{2}{11}$ **22.** $\frac{5}{99}$ **23.** $-\frac{5}{6}$

In 24–33, find a common fraction that names the same rational number as each decimal fraction.

24. 0,5 **25.** 0,555 **26.** 0,2 **27.** 0,12 **28**. 0,111

29. $0{,}125\overline{0}$ **30.** 0,2525 **31.** 0,07 **32.** 0,99875 **33.** -0,3

In Exercises 34 to 41, perform the indicated operations.

34. –7– (–15) **35.** (6–8) –12 **36.** (–2)(3–11) **37**. (7–12)(5–21)

38. 2+(–5)(–3) **39**. 4–(–4)3 **40.** (–5)(–2)(–3) **41.** (–1)+(3)(4)(–2)

In Exercises 42 to 49, use the properties of fractions to perform the indicated operations. State each answer in the lowest terms. Assume a is a nonzero real number.

42. $\frac{2a}{7}-\frac{5a}{7}$ **43.** $\frac{2a}{5}+\frac{3a}{7}$ **44.** $\frac{-3a}{5}+\frac{a}{4}$ **45.** $\frac{7}{8}a-\frac{13}{5}a$

46. $\frac{-5}{7}\cdot\frac{2}{3}$ **47.** $\frac{7}{11}\cdot\frac{-22}{21}$ **48.** $\frac{12a}{5}\div\frac{-2a}{3}$ **49.** $\frac{2}{5}\div 3\frac{2}{3}$

50. In Column I, sets of numbers are described in words. In Column II, the sets are listed using patterns and dots. Match the patterns in Column II with their correct sets in Column I.

Column I	*Column II*
1. Counting (**natural)** numbers	**a.** 0, 1, 2, . . . , 9
2. Whole numbers	**b.** 0, 1, 2, . . .
3. Even whole numbers	**c.** 0, 2, 4, 6, . . .
4. Odd whole numbers	**d.** 0, 2, 4, 6, 8
5. Even counting numbers	**e.** 0, 2, _2, 4, _4, 6, _6, . . .
6. Odd integers	**f.** 1, 2, 3, 4, . . .
7. Even integers	**g.** 1, 2, 3, . . . , 9
8. One-digit whole numbers	**h.** 1, 3, 5, 7, . . .
9. One-digit counting numbers	**i.** 1, 3, 5, 7, 9
10. Odd whole numbers less than 10	**j.** 2, 4, 6, 8, . . .
11. Even whole numbers less than 10	**k.** –2, _1, 0, 1, 2, 3, 4, . . .
12. Integers greater than _3	**l.** 1, _1, 3, _3, 5, _5, . . .

In 51–58: **a.** List all whole numbers that are factors of each of the given numbers. **b.** Is the number prime, composite, or neither?

51. 82 **52.** 101 **53.** 71 **54.** 15
55. 1 **56.** 808 **57.** 67 **58.** 397

Exercise Set 1.2

1. List all of the whole numbers that are factors of each of the given numbers.

1) 88; 2) 252; 3) 750; 4) 324; 5) 630; 6) 240.

2. Find the value of each expression

1) $\frac{11}{12}+\frac{7}{12}$ **5)** $\frac{11}{10}-\frac{1}{10}$ **9)** $-\frac{17}{21}-\frac{13}{21}$ **13)** $-\frac{15}{21}+\frac{8}{21}$ **17)** $-\frac{4}{15}+\frac{11}{15}$

2) $-\frac{10}{11}+\left(-\frac{9}{11}\right)$ **6)** $\frac{14}{23}-\frac{10}{23}$ **10)** $-\frac{1}{2}-\frac{1}{2}$ **14)** $-\frac{6}{11}+\frac{4}{11}$ **18)** $-\frac{34}{18}+\frac{67}{18}$

3) $\frac{7}{15}-\left(-\frac{3}{15}\right)$ **7)** $\frac{13}{21}+\left(-\frac{6}{21}\right)$ **11)** $-\frac{5}{7}+\left(-\frac{5}{7}\right)$ **15)** $\frac{2}{7}+\left(-\frac{5}{7}\right)$ **19)** $\frac{41}{15}+\left(-\frac{6}{15}\right)$

4) $\frac{3}{10}-\left(-\frac{7}{10}\right)$ **8)** $\frac{12}{25}+\left(-\frac{2}{25}\right)$ **12)** $\frac{6}{23}+\frac{7}{23}$ **16)** $\frac{12}{13}+\left(-\frac{14}{13}\right)$ **20)** $\frac{11}{10}+\left(-\frac{7}{10}\right)$

21) $7\frac{2}{5}+3\frac{4}{5}$ **27)** $2\frac{7}{8}-\frac{5}{8}$ **33)** $3\frac{7}{8}-4\frac{5}{8}$ **39)** $-6-\frac{2}{3}$

22) $2\frac{3}{8}+3\frac{7}{8}$ **28)** $6\frac{4}{9}-\frac{7}{9}$ **34)** $5\frac{3}{5}-6\frac{2}{5}$ **40)** $-7\frac{1}{4}-5\frac{3}{4}$

23) $1\frac{9}{11}+\frac{3}{11}$ **29)** $8\frac{2}{9}-4\frac{5}{9}$ **35)** $2\frac{5}{6}+\left(-3\frac{2}{6}\right)$ **41)** $-3\frac{7}{8}-4\frac{1}{8}$

24) $2\frac{2}{5}+\frac{1}{5}$

25) $5+\frac{3}{4}$

26) $9+\frac{7}{8}$

30) $3\frac{7}{10}-2\frac{6}{10}$

31) $2-\frac{4}{9}$

32) $6-3\frac{2}{3}$

36) $-7\frac{1}{4}+\frac{3}{4}$

37) $-5\frac{1}{3}+3\frac{2}{3}$

38) $-8+\frac{1}{4}$

42) $-9+\left(-\frac{1}{4}\right)$

43) $-2\frac{1}{3}+\left(-5\frac{1}{3}\right)$

44) $-2\frac{7}{8}+\left(-\frac{5}{8}\right)$

3. Find the value of each expression

1) $\frac{1}{4}+\frac{1}{5}$

2) $\frac{2}{3}+\frac{1}{7}$

3) $\frac{3}{5}+\frac{4}{4}$

4) $\frac{1}{2}+\frac{7}{9}$

5) $\frac{2}{3}+\frac{1}{6}$

6) $\frac{1}{2}+\frac{5}{8}$

7) $\frac{5}{6}+\frac{3}{8}$

8) $\frac{5}{8}+\frac{7}{12}$

9) $\frac{1}{16}+\frac{5}{12}$

10) $\frac{7}{20}+\frac{11}{30}$

11) $\frac{5}{48}+\frac{17}{36}$

12) $\frac{5}{24}+\frac{7}{60}$

13) $3\frac{2}{7}+5\frac{3}{14}$

14) $5\frac{1}{16}+2\frac{3}{24}$

15) $1\frac{1}{9}+2\frac{2}{5}$

16) $2\frac{2}{21}+3\frac{1}{14}$

17) $7\frac{2}{9}+4\frac{1}{6}$

18) $8\frac{3}{5}+\frac{1}{15}$

19) $7+3\frac{5}{8}$

20) $\frac{2}{3}+4\frac{3}{5}$

4. Find the value of each expression

1) $\frac{2}{3}-\frac{2}{5}$

2) $\frac{1}{2}-\frac{1}{3}$

3) $\frac{3}{5}-\frac{4}{7}$

4) $\frac{5}{7}-\frac{1}{6}$

5) $\frac{3}{4}-\frac{1}{2}$

6) $\frac{7}{10}-\frac{3}{5}$

7) $\frac{5}{9}-\frac{5}{12}$

8) $\frac{7}{12}-\frac{7}{20}$

9) $\frac{19}{21}-\frac{11}{15}$

10) $\frac{21}{22}-\frac{3}{55}$

11) $\frac{11}{21}-\frac{2}{35}$

12) $\frac{10}{63}-\frac{5}{42}$

13) $1-\frac{3}{4}$

14) $2-\frac{5}{6}$

15) $9-\frac{11}{12}$

16) $7-1\frac{7}{8}$

17) $5-2\frac{2}{5}$

18) $6-5\frac{5}{8}$

19) $8\frac{3}{11}-4$

20) $5\frac{7}{15}-\frac{3}{20}$

21) $1\frac{5}{12}-\frac{9}{10}$

22) $6\frac{3}{10}-\frac{11}{15}$

23) $5\frac{7}{8}-\frac{9}{10}$

24) $7\frac{5}{12}-3\frac{2}{9}$

25) $10\frac{1}{2}-4\frac{9}{14}$

26) $7\frac{4}{7}-5\frac{7}{9}$

27) $2\frac{3}{10}-1\frac{11}{15}$

28) $5\frac{3}{8}-3\frac{5}{6}$

5. Find the value of each expression

1) $\frac{1}{4}-\left(1-\frac{11}{12}\right)$

2) $2-\left(\frac{13}{22}-\frac{5}{22}\right)$

3) $6\frac{3}{16}-\left(2\frac{3}{8}+3\frac{5}{12}\right)$

4) $8\frac{1}{12}-3\frac{4}{15}-1\frac{7}{30}$

5) $\left(13-8\frac{5}{12}\right)+\left(17\frac{1}{2}-16\frac{1}{5}\right)$

6) $\left(63\frac{2}{3}+3\frac{1}{8}\right)-\left(13-10\frac{5}{9}\right)$

7) $\left(15\frac{1}{2}-2\frac{3}{8}\right)-\left(5\frac{5}{6}+6\frac{3}{4}\right)+\left(10\frac{2}{3}-5\frac{5}{8}\right)$

8) $\left(20-19\frac{3}{4}\right)+\left(17\frac{3}{4}-17\right)+\left(2\frac{1}{2}-\frac{17}{24}\right)$

6. Find the value of each expression

1) $\frac{3}{8}\cdot 2$
2) $\frac{5}{18}\cdot 12$
3) $\frac{7}{15}\cdot 40$
4) $\frac{7}{8}\cdot 24$
5) $\frac{1}{2}\cdot 30$
6) $\frac{9}{11}\cdot 11$
7) $\frac{2}{3}\cdot 1$
8) $\frac{19}{20}\cdot 0$

7. Find the value of each expression

1) $\frac{3}{4}\cdot\frac{5}{7}$
2) $\frac{1}{8}\cdot\frac{3}{4}$
3) $\frac{4}{7}\cdot\frac{5}{6}$
4) $\frac{2}{5}\cdot\left(-\frac{7}{11}\right)$
5) $\frac{1}{2}\cdot\left(-\frac{4}{9}\right)$
6) $\left(-\frac{11}{12}\right)\cdot\frac{8}{9}$
7) $\left(-\frac{2}{5}\right)\cdot\frac{3}{2}$
8) $\left(-\frac{11}{15}\right)\cdot\left(-\frac{3}{5}\right)$
9) $\left(-\frac{15}{16}\right)\cdot\left(-\frac{5}{9}\right)$
10) $\frac{12}{25}\cdot\frac{9}{16}$
11) $\left(-\frac{14}{17}\right)\cdot\frac{34}{63}$
12) $\frac{17}{26}\cdot\frac{13}{18}$
13) $\frac{9}{16}\cdot\left(-\frac{4}{27}\right)$
14) $\left(-\frac{1}{7}\right)\cdot\left(-\frac{49}{50}\right)$
15) $\frac{25}{72}\cdot\frac{9}{75}$

8. Find the value of each expression

1) $1\frac{2}{7}\cdot 1\frac{1}{4}$
2) $4\frac{2}{3}\cdot\frac{2}{5}$
3) $1\frac{3}{5}\cdot 3\frac{3}{4}$
4) $\frac{4}{9}\cdot 2\frac{3}{4}$
5) $2\frac{3}{4}\cdot\frac{4}{11}$
6) $1\frac{3}{4}\cdot 1\frac{5}{7}$
7) $3\frac{1}{4}\cdot 4$
8) $10\cdot 5\frac{2}{5}$
9) $3\frac{5}{6}\cdot 1\frac{7}{23}$
10) $1\frac{2}{3}\cdot 2\frac{2}{5}$
11) $7\frac{3}{11}\cdot 2\frac{19}{40}$
12) $2\frac{1}{2}\cdot 2\frac{2}{15}$
13) $0\cdot 1\frac{4}{9}$
14) $1\frac{5}{7}\cdot 1$
15) $3\frac{8}{9}\cdot 0$

9. Find the value of each expression

1) $\frac{3}{8}:\frac{5}{7}$
2) $\frac{1}{5}:\frac{3}{4}$
3) $\frac{4}{5}:\frac{4}{7}$
4) $\left(-\frac{3}{16}\right):\frac{5}{12}$
5) $\frac{3}{5}:\frac{9}{25}$
6) $\frac{7}{8}:2$
7) $\frac{3}{8}:(-3)$
8) $5:\frac{2}{5}$
9) $(-8):\frac{4}{5}$
10) $\frac{3}{7}:\left(-\frac{1}{2}\right)$
11) $3\frac{1}{2}:\frac{2}{3}$
12) $4\frac{1}{2}:\left(-1\frac{1}{2}\right)$
13) $1\frac{2}{3}:1\frac{1}{10}$
14) $\left(-10\frac{1}{3}\right):\left(-2\frac{2}{3}\right)$
15) $\frac{4}{15}:3\frac{1}{15}$
16) $4\frac{3}{4}:3$
17) $1:\frac{3}{11}$
18) $0:5\frac{1}{18}$
19) $3\frac{1}{4}:1$
20) $\left(-3\frac{7}{39}\right):\left(-1\frac{5}{31}\right)$

10. Find the value of each expression

1) $\left(\frac{5}{12}+\frac{3}{8}\right)\cdot\frac{12}{19}$
2) $\frac{6}{7}\cdot\left(\frac{11}{18}-\frac{5}{12}\right)$
3) $\left(3\frac{1}{14}-2\frac{5}{7}\right)\cdot\left(7-6\frac{3}{5}\right)$
4) $\left(3\frac{1}{12}-2\frac{3}{4}\right)\cdot\left(1\frac{1}{6}-\frac{5}{12}\right)$
5) $\left(6\frac{7}{12}-5\frac{11}{15}\right)\cdot\left(1-\frac{10}{17}-\frac{3}{17}\right)$
6) $\left(1\frac{1}{2}\right)^3-2\frac{1}{4}\cdot 1\frac{1}{3}$
7) $\left(\left(1\frac{1}{4}\right)^2-\frac{5}{8}\right)\cdot 10\frac{2}{3}-7\frac{1}{3}$
8) $\left(1\frac{4}{9}+2\frac{5}{6}-2\frac{3}{4}\right)\cdot\left(2\frac{1}{2}-\frac{11}{14}\right)$
9) $\frac{2}{3}\cdot\frac{6}{7}:\frac{4}{7}$
10) $\frac{11}{12}:\frac{7}{24}\cdot\frac{21}{22}$
11) $\left(4\frac{8}{15}-1\frac{1}{3}\right)\cdot 1\frac{7}{8}$
12) $\left(2\frac{2}{13}+1\frac{5}{6}\right):1\frac{1}{2}$
13) $\left(3\frac{1}{6}-2\frac{7}{15}\right):1\frac{2}{5}$
14) $\frac{3}{4}:\frac{5}{6}+2\frac{1}{2}\cdot\frac{2}{5}-1:1\frac{1}{6}$
15) $2\frac{3}{4}:\left(1\frac{1}{2}-\frac{2}{5}\right)+\left(\frac{3}{4}+\frac{5}{6}\right):3\frac{1}{6}$

16) $\frac{2}{3}\cdot\frac{9}{16}-\frac{5}{24}\cdot\frac{2}{5}-\frac{1}{6}$;

17) $4\frac{11}{18}\cdot\frac{6}{7}-1\frac{4}{9}$;

18) $\frac{13}{14}\cdot\frac{7}{25}:\frac{13}{25}$;

19) $\frac{15}{16}:\frac{3}{8}\cdot\frac{3}{4}$;

20) $\left(11\frac{5}{11}-8\frac{21}{22}\right):1\frac{2}{3}$;

1.4. INTERVALS, ABSOLUTE VALUE, AND DISTANCE

The real numbers can be represented geometrically by a coordinate axis called a ***real number line***. The number associated with a particular point on a real number line is called the ***coordinate*** of the point. It is customary to label those points whose coordinates are integers. The point corresponding to zero is called the ***origin***, denoted 0. Numbers to the right of the origin are ***positive real numbers***; numbers to the left of the origin are ***negative real numbers***.

Certain order relationships exist between real numbers. For example, if a and b are real numbers, then:

- a equals b (denoted by $a=b$) if $a-b=0$;
- a is greater than b (denoted by $a>b$) if $a-b$ is positive;
- a is less than b (denoted by $a<b$) if $a-b$ is negative.

The inequality symbols < and > are sometimes combined with the equality symbol in the following manner:

- $a\geq b$ read "a is greater than b or equal to b", which means $a>b$ or $a=b$;
- $a\leq b$ read "a is less than b or equal to b", which means $a<b$ or $a=b$.

Interval notation

• (a,b) represents all real numbers between a and b, not including a and not including b. This is an *open interval*.

• $[a,b]$ represents all real numbers between a and b, including a and including b. This is a *closed interval*.

• $(a,b]$ represents all real numbers between a and b, not including a but including b. This is a *half-open interval*.

• $[a,b)$ represents all real numbers between a and b, including a but not including b. This is a *half-open interval*.

• $(-\infty,a)$ represents all real numbers less than a.

• $(b,+\infty)$ represents all real numbers greater than b.

• $(-\infty,a]$ represents all real numbers less than or equal to a.

• $[b,+\infty)$ represents all real numbers greater than or equal to b.

• $(-\infty,+\infty)$ represents all real numbers .

Remark The word "or" is used to denote the union of two sets. The word "and" is used to denote the intersection of two sets.

Absolute Value and Distance

Real numbers can be represented geometrically by a ***coordinate axis*** called a ***real number line***. The number associated with a point on a real number line is called the ***coordinate of the point***. The point corresponding to zero is called the ***origin***. Every real number corresponds to a point on the number line, and every point on the number line corresponds to a real number.

If a is the coordinate of a point on the real number line, then the ***absolute value*** of a denoted by $|a|$, is the distance between a and the origin.

Definition of Absolute Value The absolute value of the real number a is defined by

$$|a| = \begin{cases} a \text{ if } a \geq 0 \\ -a \text{ if } a < 0. \end{cases}$$

EXAMPLE 12

(a) $|5| = 5$, $|-15| = 15$.

(b) Simplify $|x+2| + |x-3|$ if $-2 \leq x \leq 3$

When $-2 \leq x \leq 3$, $x-3 < 0$ and $x+2 > 0$. Therefore, $|x+2| = x+2$ and $|x-3| = -x+3$.

Thus $|x+2| + |x-3| = x+2-x+3 = 5$.

Remark The absolute value of a real number is never negative. It is always **nonnegative**, that means it is positive or zero.

The definition of distance between two points on a real number line makes use of absolute value.

Definition of the Distance Between Points on a Real Number Line If a and b are the coordinates of two points on a real number line, the distance between the graph of a and b and the graph of b is denoted by $d(a,b)$

where $d(a,b) = |a-b| = |b-a|$.

EXAMPLE 13 Find the distance between a point which coordinate on the real number line is -2 and a point whose coordinate is 5.

SOLUTION

$$d(-2,5) = |-2-5| = |5-(-2)| = 7.$$

Exercise Set 1.3

In Exercises 1 to 12, use $A = \{0,1,2,3,4\}, B = \{1,3,5,11\}$, $C = \{1,3,6,10\}$, and $D = \{0,2,4,6,8,10\}$ to find the indicated intersection or union.

1. $A \cap B$ **2.** $A \cap D$ **3.** $B \cup C$
4. $A \cap C$ **5.** $C \cap D$ **6.** $B \cup D$
7. $B \cap C$ **8.** $A \cup B$ **9.** $A \cup D$
10. $B \cap D$ **11.** $A \cup C$ **12.** $C \cup D$

In Exercises 13 to 15, graph each number on a real number line.

13. $-4; -2; \frac{7}{4}; 2.5$ **14.** $\pi; -1.2; 0.25; \frac{9}{2}$ **15.** $\frac{0}{5}; -1; -\sqrt{9}; 4.5$

In Exercises 16 to 24, replace the with the appropriate symbol (<,=, or >).

16. $\frac{5}{2} \square 4$ **17.** $\frac{2}{3} \square\ 0.6666$ **18.** $\frac{\sqrt{5}}{2} \square\ 2$ **19.** $0.\overline{36} \square\ \frac{4}{11}$
20. $-\frac{3}{2} \square -3$ **21.** $1.25 \square\ 1.3$ **22.** $0.4 \square\ \frac{4}{9}$ **23.** $\frac{0}{2} \square -\frac{0}{5}$
24. $\sqrt{5} \square\ 2$

In Exercises 24 to 27, graph each inequality and write the inequality using interval notation.

24. $3 < x < 5$ **25.** $x \geq 0$ *and* $x < 3$
26. $-2 \leq x < 1$ **27.** $x < -3$ *or* $x \geq 2$

In Exercises 28 to 33, graph each interval and write each interval as inequality.

28. $[-4, 1]$ **29.** $[2.5, \infty)$ **30.** $(-\infty, 2] \cup (3, \infty)$
31. $[-2, 3)$ **32.** $(-\infty, 3]$ **33.** $(-\infty, 1) \cup (4, \infty)$

In Exercises34 to 37, write each real number without absolute value symbols.

34. $|4|$ **35.** $|-27.4|$ **36.** $|4| - |-7|$ **37.** $|5| \cdot |-8|$

In Exercises 38 to 40, find the distance between the points whose coordinates are given.

38. 8,1 **39.** -3, 5 **40.** 16, -34

Section2. Integer and Rational Number Exponents

2.1. INTEGER EXPONENTS

Recall that if n is a natural number, then $b^n = \underbrace{b \cdot b \cdot b \cdot \ldots \cdot b}_{n-times}$. We can extend the definition of the exponent to all integers.

Definition For any nonzero real number a: $a^0 = 1$. (Any nonzero real number raised to the zero power equals 1.)

Definition If $a \neq 0$ and n is any natural number, then $a^{-n} = \frac{1}{a^n}$ *or* $\frac{1}{a^{-n}} = a^n$.

Definition If m and n are integers and $a \neq 0$, then

$$a^n \cdot a^m = a^{n+m}.$$

This property shows that multiplication of powers with like bases can be accomplished by adding exponents.

Definition If m and n are integers and $a \neq 0$, then

$$a^n : a^m = a^{n-m}$$

This property demonstrates that division of powers with like bases can be accomplished by subtracting exponents.

Definition If m and n are integers and $a \neq 0$, then $\left(a^n\right)^m = a^{n \cdot m}$.

Definition If m and n are integers and $a \neq 0$, $b \neq 0$ then

$$(a \cdot b)^n = a^n \cdot b^n \quad \text{and} \quad \left(\frac{a}{b}\right)^n = \frac{a^n}{b^n}.$$

EXAMPLE 1 Evaluate:

(a) $7^{13} \cdot 7^6$; (b) $7^{13} : 7^6$; (c) $\left(7^{13}\right)^6$; (d) 0^{13} ; (e) $\left(\frac{5}{7}\right)^{13}$; (f) 3^{-7}.

SOLUTION

(a) $7^{13} \cdot 7^6 = 7^{13+6} = 7^{19}$; (b) $7^{13} : 7^6 = 7^{13-6} = 7^7$

(c) $\left(7^{13}\right)^6 = 7^{13 \cdot 6} = 7^{78}$; (d) $0^{13} = 0$; (e) $\left(\frac{5}{7}\right)^{13} = \frac{5^{13}}{7^{13}}$; (f) $3^{-7} = \frac{1}{3^7}$.

When working with exponential expressions containing variables, we must ensure that a value of the variable does not result in an undefined expression. To simplify an expression involving exponents, write the expression in a form in which *each base occurs at most once and no powers of powers or negative exponents occur*.

EXAMPLE 2 Simplify exponential expressions:

(a) $\frac{13a^{-3}}{b^{-4}} \cdot \frac{b}{39x^{-7}}$ (b) $\frac{(2ab)^8 \cdot 8a^8b^5}{16a^5b^7}$ (c) $\frac{\left(4x^4y^3\right)^2 \cdot 64x^5y^{13}}{\left(8xy^8\right)^3}$.

SOLUTION

(a) $$\frac{13a^{-3}}{b^{-4}}\cdot\frac{b}{39x^{-7}}=\frac{13b^4}{a^3}\cdot\frac{bx^7}{39}=\frac{13b^{4+1}x^7}{39a^3}=\frac{b^5x^7}{3a^3}$$

(b) $$\frac{(2ab)^8\cdot 8a^8b^5}{16a^5b^7}=\frac{2^8a^8b^8\cdot 2^3a^8b^5}{16a^5b^7}=\frac{2^{8+3}a^{8+8}b^{8+5}}{2^4a^5b^7}=\frac{2^{11}a^{16}b^{13}}{2^4a^5b^7}=$$
$$=2^{11-4}a^{16-5}b^{13-7}=2^7a^{11}b^6$$

(c) $$\frac{\left(4x^4y^3\right)^2\cdot 64x^5y^{13}}{\left(8xy^8\right)^3}=\frac{\left(2^2\right)^2\left(x^4\right)^2\left(y^3\right)^2\cdot 2^6x^5y^{13}}{\left(2^3\right)^3x^3\left(y^8\right)^3}=\frac{2^{2\cdot 2}x^{4\cdot 2}y^{3\cdot 2}\cdot 2^6x^5y^{13}}{2^{3\cdot 3}x^3y^{8\cdot 3}}=$$
$$=\frac{2^4x^8y^6\cdot 2^6x^5y^{13}}{2^9x^3y^{24}}=\frac{2^{4+6}x^{8+5}y^{6+13}}{2^9x^3y^{24}}=2^{10-9}x^{13-3}y^{19-24}=\frac{2x^{10}}{y^5}.$$

2.2. SCIENTIFIC NOTATION

The properties of exponents provide a compact method of writing and efficient method of computing with very large or very small numbers. This method is called **scientific notation**. A number written in scientific notation has the form $a\cdot 10^n$, where n is an integer and $1\le a<10$.

For numbers greater than 10, move the decimal point to the position to the right of the first digit. The exponent n will equal the number of places the decimal point has been moved. For example, $7430000=7,43\cdot 10^6$.

For number less than 1, move the decimal point to the right of the first nonzero digit. The exponent n will be negative, and its absolute value will equal the number of places the decimal point has been moved. For example, $0,00000000431=4,31\cdot 10^{-9}$.

2.3. RATIONAL EXPONENTS AND RADICALS

The expression a^n has been defined for real numbers $a\ne 0$ and integers n. Now we wish to extend the definition of exponents to include rational numbers so that expressions as $2^{\frac{1}{2}}$ will be meaningful. Not just any definition will do. We want a definition of rational exponents for which the properties of integer exponents are true.

If the product property for exponential expressions is to hold for rational exponents, then for rational numbers p and q, $a^p\cdot a^q=a^{p+q}$. For example,

$$5^{\frac{1}{2}}\cdot 5^{\frac{1}{2}}\ \text{ must equal }\ 5^{\frac{1}{2}+\frac{1}{2}}=5^1=5.$$

Definition If n is an even positive integer and $a \geq 0$ then $a^{\frac{1}{n}}$ is the nonnegative real number such that $\left(a^{\frac{1}{n}}\right)^n = a$. If n is an odd positive integer, then $a^{\frac{1}{n}}$ is the real number such that $\left(a^{\frac{1}{n}}\right)^n = a$.

Definition For all positive integers n and m such that $\frac{m}{n}$ is in simplest form, and for all real numbers a for which $a^{\frac{1}{n}}$ is a real number, $a^{\frac{m}{n}} = \left(a^{\frac{1}{n}}\right)^m = \left(a^m\right)^{\frac{1}{n}}$.

The following exponent properties were stated earlier, but they are restated here to remind you that they have now been extended to apply to rational exponents.

Properties of Rational Exponents If p, q and r represent rational numbers and a and b are positive real numbers, then

$$a^p \cdot a^q = a^{p+q} \qquad a^p : a^q = a^{p-q} \qquad \left(a^p\right)^q = a^{p \cdot q}$$

$$\left(a^p \cdot b^q\right)^r = a^{pr} \cdot b^{qr} \qquad \left(\frac{a^p}{b^q}\right)^r = \frac{a^{pr}}{b^{qr}}$$

Let us recall that an exponential expression is in simplest form when no powers of powers or negative exponents occur and each base occurs at most once.

EXAMPLE 3 Simplify exponential expressions:

(a) $64^{\frac{2}{3}}$ (b) $32^{-\frac{2}{5}}$ (c) $\dfrac{\left(x^{\frac{3}{4}}y^{\frac{3}{5}}\right)^2}{\left(x^{\frac{1}{3}}y^{\frac{3}{4}}\right)^3}$.

SOLUTION

(a) $64^{\frac{2}{3}} = \left(64^{\frac{1}{3}}\right)^2 = 4^2 = 16$;

(b) $32^{-\frac{2}{5}} = \left(32^{\frac{1}{5}}\right)^{-2} = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}$;

(c) $$\frac{\left(x^{\frac{3}{4}}y^{\frac{3}{5}}\right)^2}{\left(x^{\frac{1}{3}}y^{\frac{3}{4}}\right)^3}=\frac{x^{\frac{3}{4}\cdot 2}y^{\frac{3}{5}\cdot 2}}{x^{\frac{1}{3}\cdot 3}y^{\frac{3}{4}\cdot 3}}=\frac{x^{\frac{3}{2}}y^{\frac{6}{5}}}{x^1y^{\frac{9}{4}}}=x^{\frac{3}{2}-1}y^{\frac{6}{5}-\frac{9}{4}}=x^{\frac{1}{2}}y^{-\frac{21}{20}}=\frac{x^{\frac{1}{2}}}{y^{\frac{21}{20}}}.$$

Exercise Set 2.1

In Exercises 1 to 15, evaluate each expression.

1. $(-4)^3$

2. -4^3

3. 7^0

4. $(-1)^{18}$

5. $3^2\cdot 3^3$

6. $\frac{3^{-1}}{3^2}$

7. $2^7\cdot 2^{-3}\cdot 2$

8. $\frac{4^{-8}}{4^{-11}}$

9. $\left(\frac{5^{-3}\cdot 7}{3^{-2}}\right)^{-1}$

10. $\left(\frac{4\cdot 5^{-1}}{2^{-3}}\right)^{-2}$

11. $\left(\frac{4}{9}\right)^{-2}$

12. $\left(\frac{3^{-2}}{2^{-1}\cdot 5}\right)^2$

13. $\frac{(2\cdot 5)^2}{(2^{-1}\cdot 5)^3}$

14. $\frac{(2^2\cdot 3^{-1})^3}{(3\cdot 5)}$

15. $\left(\frac{-3^6\cdot 2^{-4}}{-4^{-5}}\right)^0$

In Exercises 16 to 24, simplify each exponential expression so that all exponents are positive.

16. $(2x^2y^3)(3x^5y)$

17. $\left(\frac{2ab^2c^3}{5ab^2}\right)^3$

18. $\frac{(3xy^{-3})^2}{(2xy)^{-2}}$

19. $(2x^{-3}y^0)(3^{-1}xy)^2$

20. $\left(\frac{2x}{5y}\right)^{-2}$

21. $(x^2y^{-3})^{-2}$

22. $a^{-1}+b^{-2}$

23. $\frac{4a^2(bc)^{-1}}{(-2)^2a^3b^{-2}c}$

24. $(2ab^{-3})^2(-2a^{-1}b^3)^2$

In Exercises 25 to 30, write each number in scientific notation.

25. 73.4 **27.** 21000000 **28.** 521 **29.** 0.00000714 **30**. 0.00095

26. 25600

In Exercises 31 to 35, change each number from its scientific notation to its decimal form.

31. $6.5\cdot 10^3$ **33.** $8.0\cdot 10^{10}$ **34.** $4.007\cdot 10^{-3}$ **35.** $3.75\cdot 10^0$

32. $7.31\cdot 10^{-5}$

In Exercises 36 to 39, write each expression as an equivalent expression in which the variables x and y occur only once. All the exponents are integers.

36. $\left(\frac{x^{3n}y^{2n}}{x^{-2n}y^{3n+1}}\right)^{-1}$ **37.** $\frac{x^ny^{n+2}}{x^{n-3}y}$ **38.** $\left(\frac{x^ny}{x^{1-n}y^{-1}}\right)^2$

In Exercises 39 to 48, evaluate each expression

39. $16^{\frac{3}{4}}$ **40.** $25^{\frac{3}{2}}$ **41.** $81^{\frac{3}{4}}$ **42.** $125^{\frac{2}{3}}$ **43.** $36^{\frac{3}{2}}$

44. $64^{-\frac{5}{6}}$ **45.** $128^{-\frac{5}{6}}$ **46.** $49^{-\frac{3}{2}}$ **47.** $1024^{-\frac{3}{10}}$ **48.** $32^{-\frac{3}{5}}$

In Exercises 49 to 51, simplify each exponential expression so that all exponents are positive.

49. $\dfrac{\left(-2x^{\frac{3}{5}}y^{\frac{4}{7}}\right)^2}{\left(3x^{-\frac{2}{3}}y^{\frac{3}{4}}\right)^3}$ **50.** $\dfrac{\left(3x^{\frac{3}{2}}y^{-\frac{3}{5}}z^{\frac{1}{2}}\right)^2}{\left(-2x^{\frac{1}{3}}y^{-\frac{3}{4}}z^{\frac{1}{3}}\right)^3}$ **51.** $\dfrac{\left(4x^2y^{\frac{8}{5}}\right)^{\frac{1}{2}}}{\left(-27x^{\frac{1}{3}}y^{\frac{3}{4}}\right)^{\frac{1}{3}}}$

2.4. SIMPLIFYING RADICAL EXPRESSIONS

Radicals, expressed by the notation $\sqrt[n]{a}$ are also used to denote roots. The number a is the **radicand**, and the positive integer n is the **index** of the radical.

> ***Definition*** If n is a positive integer and a is a real number such that $a^{\frac{1}{n}}$ is a real number, then $\sqrt[n]{a} = a^{\frac{1}{n}}$.

If the index n equals 2, then the radical $\sqrt[2]{a}$ is written as simply $\sqrt{a}$ and it is referred to as the **principal square root** of a, or simply the square root of a.

The symbol $\sqrt{a}$ is reserved to represent the nonnegative square root of a. To represent the negative square root of a, write $-\sqrt{a}$. For example, $\sqrt{25} = 5$, whereas $-\sqrt{25} = -5$.

> ***Definition*** For all positive integers n, all integers m and all real numbers a such that $\sqrt[n]{a}$ is a real number, $\left(\sqrt[n]{a}\right)^m = \sqrt[n]{a^m} = a^{\frac{m}{n}}$.

> ***Definition*** If n is an ***even*** natural number and a is a real number, then
> $$\sqrt[n]{a^n} = |a|$$
> If n is an ***odd*** natural number and a is a real number, then
> $$\sqrt[n]{a^n} = |a|$$
> $$\sqrt[n]{a^n} = a$$

Because radicals are defined in terms of rational powers, the properties of radicals are similar to those of exponential expressions.

Properties of Radicals If m, n represent rational numbers and a and b are positive real numbers, then

$\sqrt[n]{a}\cdot\sqrt[n]{b}=\sqrt[n]{ab}$ $\qquad$ $\dfrac{\sqrt[n]{a}}{\sqrt[n]{b}}=\sqrt[n]{\dfrac{a}{b}}$ $\qquad$ $\sqrt[n]{\sqrt[m]{a}}=\sqrt[nm]{a}$

A radical is in the **simplest form** if it meets all of the following criteria.

1. The radicand contains only powers less than the index ($\sqrt{x^5}$ does not satisfy this requirement because 5, the exponent, is greater than 2, the index).

2. The index of the radical is as small as possible ($\sqrt[9]{x^3}$ does not satisfy this requirement because $\sqrt[9]{x^3}=x^{\frac{3}{9}}=x^{\frac{1}{3}}=\sqrt[3]{x}$).

3. The denominator has been rationalized. That is, no radicals occur in the denominator ($\dfrac{1}{\sqrt{2}}$ does not satisfy this requirement).

4. No fractions occur under the radical sign ($\sqrt[4]{\dfrac{2}{x^3}}$ does not satisfy this requirement).

Radical expressions are simplified by using the properties of radicals. **Like radicals** have the same radicand and the same index. Addition and subtraction of like radicals are accomplished by using the distributive property. Sometimes it is possible to simplify radical expressions that do not appear to be like radicals by simplifying each radical expression. Here are some examples.

EXAMPLE 4 Simplify:

(a) $\sqrt[4]{16x^4}$ (b) $\sqrt[5]{32y^5}$ (c) $\sqrt[4]{32x^4y^5}$ (d) $5x\sqrt[3]{16x^4}-\sqrt[3]{128x^7}$

SOLUTION

(a) $\sqrt[4]{16x^4}=\sqrt[4]{2^4x^4}=2|x|$;

(b) $\sqrt[5]{32y^5}=\sqrt[5]{2^5y^5}=2y$;

(c) $\sqrt[4]{32x^4y^5}=\sqrt[4]{2^4\cdot 2x^4y^4\cdot y}=2y|x|\sqrt[4]{2y}$,

(d) $5x\sqrt[3]{16x^4}-\sqrt[3]{128x^7}=5x\sqrt[3]{2^3\cdot 2\cdot x^3\cdot x}-\sqrt[3]{2\cdot 2^6\cdot x^6\cdot x}=$

$=5x\cdot 2x\sqrt[3]{2x}-2^2\cdot x^2\sqrt[3]{2x}=10x^2\sqrt[3]{2x}-4x^2\sqrt[3]{2x}=6x^2\sqrt[3]{2x}$.

Multiplication of radical expressions is accomplished by using the distributive property. Finding the product of more complicated radical expressions may require repeatitive use of the distributive property.

EXAMPLE 5 Perform the indicated operation

(a) $\left(2\sqrt{3}-3\right)\left(4-5\sqrt{3}\right)$ (b) $\left(3-\sqrt{x-5}\right)^2$, $x\le 5$.

SOLUTION

(a) $\left(2\sqrt{3}-3\right)\left(4-5\sqrt{3}\right)=2\sqrt{3}\cdot 4-2\sqrt{3}\cdot 5\sqrt{3}-3\cdot 4+3\cdot 5\sqrt{3}=$

$=8\sqrt{3}-10\cdot 3-12+15\sqrt{3}=-42+23\sqrt{3}$.

(b)

$$\left(3-\sqrt{x-5}\right)^2=\left(3-\sqrt{x-5}\right)\cdot\left(3-\sqrt{x-5}\right)=3\cdot 3-3\cdot\sqrt{x-5}-3\cdot\sqrt{x-5}+\left(\sqrt{x-5}\right)^2=$$

$$=\left[\left(\sqrt{x-5}\right)^2=x-5\ ,\ x\geq 5\ \right]=9-6\sqrt{x-5}+x-5=x+4-6\sqrt{x-5}\ .$$

To **rationalize the denominator** of a fraction means to write the fraction in an equivalent form that does not involve any radicals in the denominator. It is accomplished by multiplying the numerator and denominator of the radical expression by an expression that will cause the radicand in the denominator to be a perfect root of the index.

EXAMPLE 6 Rationalize the denominator

(a) $\dfrac{7}{\sqrt{2}}$ (b) $\dfrac{3}{\sqrt[3]{5}}$ (c) $\dfrac{7}{\sqrt{3x}}$ (d) $\dfrac{2}{\sqrt[5]{3y^3}}$.

SOLUTION

(a) $$\frac{7}{\sqrt{2}}=\frac{7}{\sqrt{2}}\cdot\frac{\sqrt{2}}{\sqrt{2}}=\frac{7\sqrt{2}}{\sqrt{2^2}}=\frac{7\sqrt{2}}{2}\quad ;$$

(b) $$\frac{3}{\sqrt[3]{5}}=\frac{3}{\sqrt[3]{5}}\cdot\frac{\sqrt[3]{5^2}}{\sqrt[3]{5^2}}=\frac{3\sqrt[3]{5^2}}{\sqrt[3]{5^3}}=\frac{3\sqrt[3]{5^2}}{5}\quad ;$$

(c) $$\frac{7}{\sqrt{3x}}=\frac{7}{\sqrt{3x}}\cdot\frac{\sqrt{3x}}{\sqrt{3x}}=\frac{7\sqrt{3x}}{\sqrt{(3x)^2}}=\frac{7\sqrt{3x}}{3x}\quad ,\ if\quad x>0\quad ;$$

(d) $$\frac{2}{\sqrt[5]{3y^3}}=\frac{2}{\sqrt[5]{3y^3}}\cdot\frac{\sqrt[5]{(3y^3)^4}}{\sqrt[5]{(3y^3)^4}}=\frac{2\sqrt[5]{(3y^3)^4}}{\sqrt[5]{(3y^3)^5}}=\frac{2\sqrt[5]{(3y^3)^4}}{3y^3}\ .$$

To rationalize the denominator of a fractional expression such as $\dfrac{1}{\sqrt{m}+\sqrt{n}}$ we use the conjugate of $\left(\sqrt{m}+\sqrt{n}\right)$ which is $\left(\sqrt{m}-\sqrt{n}\right)$. The product of these conjugate pairs does not involve a radical.

$$\left(\sqrt{m}+\sqrt{n}\right)\left(\sqrt{m}-\sqrt{n}\right)=m-n$$

EXAMPLE 7 Rationalize the denominator

(a) $\dfrac{3+2\sqrt{5}}{1-\sqrt{5}}$ (b) $\dfrac{3-\sqrt{x}}{4+3\sqrt{x}}$, $if\ x>0$.

SOLUTION

(a) $\dfrac{3+2\sqrt{5}}{1-\sqrt{5}}=\dfrac{3+2\sqrt{5}}{1-\sqrt{5}}\cdot\dfrac{1+\sqrt{5}}{1+\sqrt{5}}=\dfrac{\left(3+2\sqrt{5}\right)\left(1+\sqrt{5}\right)}{1-5}=\dfrac{13+5\sqrt{5}}{-4}=-\dfrac{13+5\sqrt{5}}{4}$;

(b) $\dfrac{3-\sqrt{x}}{4+3\sqrt{x}}=\dfrac{3-\sqrt{x}}{4+3\sqrt{x}}\cdot\dfrac{4-3\sqrt{x}}{4-3\sqrt{x}}=\dfrac{\left(3-\sqrt{x}\right)\left(4-3\sqrt{x}\right)}{4^2-\left(3\sqrt{x}\right)^2}=$

$=\dfrac{12-4\sqrt{x}-9\sqrt{x}+3x}{16-9x}=\dfrac{12+3x-13\sqrt{x}}{16-9x}$.

Exercise Set 2.2

In Exercises 1 to 12, evaluate each expression

1. $4\sqrt{2}-\sqrt{18}$ **2.** $\sqrt{125}-7\sqrt{5}$ **3.** $\left(\sqrt{12}+\sqrt{27}\right)\sqrt{3}$

4. $\sqrt[3]{0{,}1^3\cdot 20^6}$ **5.** $\dfrac{5\sqrt[3]{17}}{\sqrt[3]{136}}$ **6.** $\sqrt{5^2-4^2}$

7. $\sqrt{65^2-63^2}$ **8.** $\left(\sqrt{5}-\sqrt{3}\right)^2+2\sqrt{15}$ **9.** $\left(3\sqrt{2}+2\sqrt{3}\right)^2-\left(3\sqrt{2}+2\sqrt{3}\right)\left(3\sqrt{2}-2\sqrt{3}\right)$

10. $\dfrac{\sqrt[9]{7}\cdot\sqrt[18]{7}}{\sqrt[6]{7}}$ **11.** $\sqrt{(\sqrt{15}-3)^2}$ **12.** $\sqrt{(\sqrt{17}-8)^2}$

In Exercises 13 to 20, rationalize the denominator

13. $\dfrac{3}{\sqrt{6}}$ **14.** $-\dfrac{5}{\sqrt{7}}$

15. $\dfrac{1}{\sqrt{2}-1}$ **16.** $\dfrac{9}{\sqrt{7}+\sqrt{5}}$

17. $\left(\dfrac{5}{\sqrt{3}+2}-\dfrac{8}{\sqrt{5}-\sqrt{3}}\right)\left(\sqrt{3}+6\right)$ **18.** $\left(\dfrac{16}{\sqrt{5}-1}-\dfrac{5}{\sqrt{3}+2}\right)\left(\sqrt{3}+2\right)$

19. $\dfrac{1-2\sqrt{x}}{3-4\sqrt{x}}$ **20.** $\dfrac{3+2\sqrt{x}}{4-\sqrt{x}}$

In Exercises 21 to 26, simplify

21. $\sqrt[4]{81x^4}$ *if* $x<0$ **22.** $\sqrt[5]{243y^5}$ **23.** $\sqrt[6]{64x^6y^9}$ *if* $x<0$

24. $2x\sqrt[3]{8x^4}-3\sqrt[3]{27x^7}$ **25.** $5z\sqrt[4]{16z^5}-\sqrt[4]{81z^9}$ **26.** $2\sqrt[3]{16x^4}-x\sqrt[3]{54x}$

Section 3. POLYNOMIALS

3.1. POLYNOMIALS

A **monomial** is a constant, or a variable, or a product of a constant and one or more variables, with the variables having only nonnegative integer exponents. The constant is called the **numerical coefficient** or simply the **coefficient** of the monomial. The **degree** of the monomial is the sum of the exponents of the variables. For example,

- $-5xy^2$ is a monomial with coefficient –5 and degree 3;
- $\frac{1}{2}u$ is a monomial with coefficient $\frac{1}{2}$ and degree 1;
- 0.25 is a monomial with coefficient 0.25 and degree 0.

The algebraic expressions $3x^{-2}$ and $\frac{5}{x}$ are not monomials because they cannot be written as a product of a constant and a variable with a nonnegative integer exponent.

A sum of a finite number of monomials is called a **polynomial**. Each monomial is called a **term of the polynomial**. The **degree of a polynomial** is the largest degree of the terms in the polynomial.

Terms of polynomials that have exactly the same variables raised to the same powers are called **like terms**. For example,

- $14x^2$ and $-31x^2$ are like terms;
- $3x^2y$ and $7xy$ are not like terms because x^2y and xy are not identical;
- $5x^2y^3$ and $-4y^3x^2$ are like terms because $y^3x^2 = x^2y^3$ by the commutative property of multiplication.

A polynomial is said to be simplified if all its like terms have been combined. For example, the simplified form $4x^2+6x-8x$ is $4x^2-3x$.

A simplified polynomial that has two terms is called **binomial**, and a simplified polynomial that has three terms is called a **trinomial**. For example,

- $5x-3$ is binomial of degree 1;
- $5x^3-4x^2+5$ is trinomial of degree 3;
- $5x^3-4x^2y^3-7x^4+8$ is a polynomial of degree 5.

A nonzero constant, such as 5, is called a **constant polynomial**. It has degree zero since $5 = 5x^0$. The number 0 is defined be a polynomial with no degree.

General Form of a Polynomial

The **general form of polynomial** of degree n in the single variable x is

$$a_nx^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0x$$

Where $a_n \neq 0$ and n is a nonnegative integer. The coefficient a_n is called the **leading coefficient**

If a polynomial in the single variable x is written with decreasing powers of x, then it is in a **standard form**. For example, the polynomial $3x^2-4x^3+7x^4-5$ is written in standard form as $7x^4-4x^3+3x^2-5$.

Remark. Because the polynomial is defined as the sum of its terms, the coefficient of each term includes the sign between the terms. For instance, the polynomial $3x^2-5x+4$ has terms of $3x^2$, $-5x$, 4. The coefficients of the terms are $3, -5, 4$.

The following table shows the leading coefficient, degree, terms, and coefficients of the given polynomials:

Polynomial	**Leading Coefficient**	**Degree**	**Terms**	**Coefficients**
$-6x^2+3x-4$	-6	2	$-6x^2, 3x, -4$	$-6, 3, -4$
$3x-5$	3	1	$3x, -5$	$3, -5$
x^3-x+3	1	3	$x^3, -x, 3$	$1, -1, 3$

To add polynomials, we add like terms.

EXAMPLE 1 **(Add polynomials)** Find the sum $\left(3x^2-4x+5\right)+\left(-5x^2+8x-3\right)$.

SOLUTION

$$\left(3x^2-4x+5\right)+\left(-5x^2+8x-3\right)=\left(3x^2-5x^2\right)+\left(-4x+8x\right)+\left(5-3\right)=-2x^2+4x+2.$$

To subtract polynomials, we subtract like terms.

EXAMPLE 2 (Subtract polynomials) Find the difference.

$$\left(x^2+7x-3\right)-\left(-3x^2+8x-5\right)$$

SOLUTION

$$\left(x^2+7x-3\right)-\left(-3x^2+8x-5\right)=\left(x^2-\left(-3x^2\right)\right)+\left(7x-8x\right)+\left(-3-(-5)\right)=$$
$$=\left(x^2+3x^2\right)+\left(-x\right)+\left(-3+5\right)=4x^2-x+2.$$

EXAMPLE 3 **(Multiply polynomials)** Find $\left(4x-3\right)\cdot\left(8x+2\right)$.

SOLUTION

$$\left(4x-3\right)\cdot\left(8x+2\right)=\left(4x\right)\left(8x\right)+\left(4x\right)\left(2\right)+\left(-3\right)\left(8x\right)+\left(-3\right)\left(2\right)=$$
$$=32x^2+8x-24x-6=32x^2+\left(8x-24x\right)-6=32x^2-16x-6.$$

EXAMPLE 4 (**Multiply polynomials)** Find $\left(3x-4\right)\cdot\left(2x^2+5x+1\right)$.

SOLUTION

$$\left(3x-4\right)\cdot\left(2x^2+5x+1\right)=\left(3x\right)\left(2x^2\right)+\left(3x\right)\left(5x\right)+\left(3x\right)\left(1\right)+\left(-4\right)\left(2x^2\right)+$$
$$+\left(-4\right)\left(5x\right)+\left(-4\right)\left(1\right)=6x^3+15x^2+3x-8x^2-20x-4=6x^3+\left(15x^2-8x^2\right)+$$
$$+\left(3x-20x\right)-4=6x^3+7x^2-17x-4.$$

3.2. FACTORING

Writing the polynomial as a product of polynomials of lower degree is called **factoring**. Factoring is an important procedure, often used to simplify fractional expressions and solve equations.

EXAMPLE 1 (**Factor out the Greatest Common Factor)** Factor out the GCF in each of the following:

(a) $8x^2-4x$ (b) $6x-12$ (c) $27x^3+18x^2$.

SOLUTION

(a)

$8x^2-4x=(4x)(2x)-(4x)(1)=$	*Factor each term.*
$=4x(2x-1)$	*Factor out the GCF*

(b)

$6x-12=(6)(x)-(6)(2)=$	*Factor each term.*
$=6(x-2)$	*Factor out the GCF*

(c)

$27x^3+18x^2=(9x^2)(3x)+(9x^2)(2)=$	*Factor each term.*
$=9x^2(3x+2)$	*Factor out the GCF*

Some polynomials can be **factored by grouping**. Pairs of terms that have a common factor are first grouped together.

EXAMPLE 2 ***(*Factor by grouping in pairs)** Factor $6y^3-21y^2-4y+14$.

SOLUTION.

$6y^3-21y^2-4y+14=$	
$=(6y^3-21y^2)+(-4y+14)=$	*Group the terms in pairs.*
$=3y^2(2y-7)-2(2y-7)=$	*Factor out the common monomial factors.*
$=(2y-7)(3y^2-2)$	*Factor out the common binomial factor.*

Some polynomials can be factored by using special product formulas in reverse. The following factoring formulas can be verified by multiplying the factors on the right side of each equation.

Factoring Formulas

Difference of two squares	$x^2 - y^2 = (x-y)(x+y)$
Perfect square trinomials	$x^2 + 2xy + y^2 = (x+y)^2$ $x^2 - 2xy + y^2 = (x-y)^2$
Sum of cubes	$x^3 + y^3 = (x+y)(x^2 - xy + y^2)$
Difference of cubes	$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$
Cube of sum	$x^3 + 3x^2y + 3xy^2 + y^3 = (x+y)^3$
Cube of difference	$x^3 - 3x^2y + 3xy^2 - y^3 = (x-y)^3$

EXAMPLE 3 **(Factor the difference of Squares)**

Factor (a) $49y^2 - 4$ (b) $144a^2 - 1$.

SOLUTION

(a)

$49y^2 - 4 = (7y)^2 - (2)^2 =$	*Recognize the difference of squares form*
$= (7y-2)(7y+2)$	*The binomial factors are the sum and the difference of the square roots of squares.*

(b)

$144a^2 - 1 = (12a)^2 - (1)^2 =$	*Recognize the difference of squares form*
$= (12a-1)(12a+1)$	*Factor.*

EXAMPLE 4 (**Factor Perfect Square Trinomials)**

Factor (a) $4a^2 + 12a + 9$ (b) $16m^2 - 40mn + 25n^2$.

SOLUTION

(a)

$$4a^2 + 12a + 9 = (2a)^2 + 2 \cdot 2a \cdot 3 + 3^2 = \left[\begin{array}{l} \textit{use the factoring formula} \\ x^2 + 2xy + y^2 = (x+y)^2, \textit{ where } x = 2a,\ y = 3 \end{array}\right] =$$

$$= (2a+3)^2$$

(b)

$$16m^2 - 40mn + 25n^2 = (4m)^2 - 2\cdot 4m\cdot 5n + (5n)^2 = \begin{bmatrix} \textit{use the factoring formula} \\ x^2 - 2xy + y^2 = (x-y)^2, \textit{ where } x = 4m,\ y = 5n \end{bmatrix} =$$

$$= (4m - 5n)^2$$

***EXAMPLE 5 (*Factor the Sum of Difference of Cubes)**

Factor (a) $8a^3 + b^3$ (b) $m^3 - 64$.

SOLUTION

(a)

$8a^3 + b^3 = (2a)^3 + (b)^3 =$	*Recognize the sum of cubes form*
$= (2a+b)\left((2a)^2 - (2a)\cdot(b) + (b)^2\right) =$	*Factor.*
$= (2a+b)\left(4a^2 - 2ab + b^2\right)$	

(b)

$m^3 - 64 = (m)^3 - (4)^3 =$	*Recognize the difference of cubes form*
$= (m-4)\left((m)^2 + (m)(4) + (4)^2\right) =$	*Factor.*
$= (m-4)\left(m^2 + 4m + 16^2\right)$	

EXAMPLE 6 Factor completely each of the following polynomials:

(a) $32x^4 - 162$; (b) $x^6 + 7x^3 - 8$.

SOLUTION

(a)

$32x^4 - 162 = 2\left(16x^4 - 81\right) =$	*Factor out the common factor.*
$= 2\left(\left(4x^2\right)^2 - 9^2\right) =$	*Recognize the difference of squares form.*
$= 2\left(4x^2 + 9\right)\left(4x^2 - 9\right) =$	*Factor the difference of squares form*
$= 2\left(4x^2 + 9\right)(2x-3)(2x+3)$	*Factor the difference of squares.*

(b)

$$x^6 + 7x^3 - 8 = x^6 + \left(7x^3\right) - 8 = x^6 + \left(8x^3 - x^3\right) - 8 = x^6 + 8x^3 - x^3 - 8 =$$

$$= \left(x^6 - x^3\right) + \left(8x^3 - 8\right) = x^3\left(x^3 - 1\right) + 8\left(x^3 - 1\right) = \left(x^3 - 1\right)\left(x^3 + 8\right)$$

$$= (x-1)\left(x^2 + x + 1\right)(x+2)\left(x^2 - 2x + 4\right).$$

When factoring by grouping, it may not be clear which terms should be grouped. Some experimentation may be necessary to find a grouping that is in the form of the special factoring formulas.

***EXAMPLE 7* (Factor by Grouping)**

Use the technique of grouping to factor each of the following:
(a) $a^2+10ab+25b^2-9$; (b) $a^2+a-b-b^2$.

SOLUTION

(a)

$a^2+10ab+25b^2-9=$

$=\left(a^2+10ab+25b^2\right)-3^2=$	*Group the terms of the perfect square trinomial*
$=(a+5b)^2-3^2=$	*Factor the trinomial*
$=\left((a+5b)-3\right)\left((a+5b)+3\right)=$	*Factor the difference of squares.*
$=(a+5b-3)(a+5b+3)$	*Simplify.*

(b)

$a^2+a-b-b^2=$

$=a^2-b^2+a-b=$	*Rearrange the terms*
$=\left(a^2-b^2\right)+(a-b)=$	*Regroup.*
$=(a-b)(a+b)+(a-b)=$	*Factor the difference of squares.*
$=(a-b)(a+b+1)$	*Factor out the common factor $(a-b)$.*

3.3. FACTOR TRINOMIALS

To factor trinomials ax^2+bx+c, where $a,b,c\in R$ and $a\neq 0$ it is necessary to solve quadratic equation $ax^2+bx+c=0$.

The formula for solving quadratic equation $ax^2+bx+c=0$ is known as the **quadratic formula.**

The Quadratic Formula

If $ax^2+bx+c=0$, $a\neq 0$, then:
1. find discriminant denoted by D: $D=b^2-4ac$,
2. find the roots of quadratic equation: $x_1=\dfrac{-b-\sqrt{D}}{2a}$, $x_2=\dfrac{-b+\sqrt{D}}{2a}$

Factor Trinomial

$$ax^2+bx+c=a(x-x_1)(x-x_2),$$

where x_1, x_2 are the roots of quadratic equation $ax^2+bx+c=0$.

EXAMPLE 8 **(Factor Trinomials)** Factor $6x^2-11x+4$.

SOLUTION

1. Find the roots of quadratic equation $6x^2-11x+4=0$:

$$D=b^2-4ac=\begin{bmatrix} a=6, \\ b=-11, c=4 \end{bmatrix}=(-11)^2-4\cdot 6\cdot 4=121-96=25, \sqrt{25}=5,$$

$$x_1=\frac{-b+\sqrt{D}}{2a}=\frac{-(-11)+5}{2\cdot 6}=\frac{11+5}{12}=\frac{16}{12}=\frac{4}{3},$$

$$x_2=\frac{-b-\sqrt{D}}{2a}=\frac{-(-11)-5}{2\cdot 6}=\frac{11-5}{12}=\frac{6}{12}=\frac{1}{2},$$

2. Factor:

$$6x^2-11x+4=6\left(x-\frac{4}{3}\right)\left(x-\frac{1}{2}\right)=2\cdot 3\cdot\left(x-\frac{4}{3}\right)\left(x-\frac{1}{2}\right)=2\cdot\left(x-\frac{1}{2}\right)\cdot 3\cdot\left(x-\frac{4}{3}\right)=$$
$$=(2x-1)(3x-4)$$

Exercise Set 3.1

In Exercises 1 to 4, for each polynomial determine its: **a.** standard form, **b.** degree, **c.** coefficients, **d.** leading coefficient, **e.** terms.

1. $2x+x^2-7$ **2.** x^3-1 **3.** $4x^2-2x+7$ **4.** $3x^2-5x^3+7x-1$

In Exercises 5 to 7, determine the degree of the given polynomial.

5. $3xy^2-2xy+7x$ **6.** $x^3+3x^2y+3xy^2+y^3$ **7.** $4x^2y^2-5x^3y^2+17xy^3$

In Exercises 8 to 13, perform the indicated operations and simplify if possible by combining like terms. Write the result in standard form.

8. $(3x^2+4x+5)+(2x^2+7x-2)$

9. $(4w^3-2w+7)+(5w^3+8w^2-1)$

10. $(r^2-2r-5)-(3r^2-5r+7)$

11. $(u^3-3u^2-4u+8)-(u^3-2u+4)$

12. $(4x-5)(2x^2+7x-8)$

13. $(3x^2-2x+5)(2x^2-5x+2)$

In Exercises 14 to 22 find the indicated product.

14. (2x+4)(5x+1)

15. (y+2)(y+1)

16. (4z-3)(z-4)

17. (a+6)(a-3)

18. (b-4)(b+6)

19. (5x-11y)(2x-7y)

20. (9x+5y)(2x+5y)

21. (6w-11x)(2w-3x)

22. (3p-5q)(2p-7q)

In Exercises 23 to 25, perform the indicated operations and simplify.

23. $(4d-1)^2-(2d-3)^2$ **24.** $(r+s)(r^2-rs+s^2)$ **25.** $(3c-2)(4c+1)(5c-2)$

In Exercises 26 to 30, use the special product formulas to perform the indicated operation.

26. $(3x+5)(3x-5)$ **28.** $(4w+z)^2$ **30.** $[(x+5)+y][(x+5)-y]$

27. $(3x^2-y)^2$ **29.** $[(x-2)+y]^2$

In Exercises 31 to 33, evaluate the given polynomial for the indicated value of the variable.

31. x^2+7x-1**,** for x = 3 **32.** $-x^2+5x-3$, for x = -2

33. $3x^3-2x^2-x+3$, for x = -1

In Exercises 34 to 42, factor out the GCF from each polynomial.

34. $5x+20$ **37.** $10x^2y+6xy-14xy^2$ **40.** $ax^2-ax+bx-b$

35. $8x^2+12x-40$ **38.** $(x-3)(a+b)+(x-3)(a+3b)$ **41.** $6w^3+4w^2-15w-10$

36. $-6y^2-54y$ **39.** $3x^3+x^2+6x+2$ **42.** $12a^2x^3-8abx-15ax^2+10b$

In Exercises 43 to 47, factor each trinomial.

43. $x^2+7x+12$ **45.** $8a^2-26a+15$ **47.** $b^2+12b-28$

44. $a^2-10a-24$ **46.** $51x^2-5x-4$

In Exercises 33 to 42, factor each difference of squares.

33. x^2-9 **34.** x^2-64 **35.** $4a^2-49$ **36.** $81b^2-16c^2$ **37.** $1-100x^2$

In Exercises 43 to 50, factor each perfect square trinomial.

43. $x^2+10x+25$ **44.** y^2+6y+9 **45.** $a^2+14a+49$ **46.** $b^2-24b+144$

In Exercises 51 to 58, factor each sum or difference of cubes.

51. x^3-8 **52.** b^3+64 **53.** $8x^3-27y^3$ **54.** $64u^3-27v^3$

In Exercises 55 to 69, factor each polynomial.

55. $18x^2-2$ **58.** $6ax^2-19axy-20ay^2$ **62.** $64y^3-16y^2z+yz^2$ **66.** $8x^2+3x-4$

56. $16x^4-1$ **59.** $3bx^3+4bx^2-3bx-4b$ **63.** $5xy+20y-15x-60$ **67.** $16x^2+81$

57. $81y^4-16$ **60.** $72bx^2+24bxy+2by^2$ **64.** $x^2+6xy+9y^2-1$ **68.** $3(y+4)^3-81$

61. $5x(2x-5)^2-(2x-5)^3$ **65.** $4x^2+2x-y-y^2$ **69.** $a^2+a+b-b^2$

Supplemental Exercises

In Exercises 70 to 75, find the indicated products.

70. $(a+b)^3$ **71.** $(a-b)^3$ **72.** $(x-1)^3$ **73.** $(y+2)^3$ **74.** $(2x-3y)^3$ **75.** $(3x+5y)^3$

3.4. RATIONAL EXPRESSIONS

A **rational expression** is a faction in which the numerator and denominator are polynomials. For example, $\frac{3}{x+1}$ and $\frac{x^2-4x-21}{x^2-9}$ are rational expressions.

However, $\dfrac{5x+2}{\sqrt{x}-3}$ is not a rational expression because the denominator is not a polynomial.

Rational expressions have properties similar to the properties of rational numbers.

Properties of Rational Expressions

For all rational expressions *P/Q* and *R/S* where $Q \neq 0$ and $S \neq 0$:

1.	***Equality***	$\dfrac{P}{Q}=\dfrac{R}{S}$ if and only if $PS=QR$
2.	***Equivalent expressions***	$\dfrac{P}{Q}=\dfrac{PR}{QR}, \quad R \neq 0$
3.	***Sign properties***	$-\dfrac{P}{Q}=\dfrac{-P}{Q}=\dfrac{P}{-Q}$

To **simplify a rational expression,** factor the numerator and the denominator. Then use the equivalent expressions property to eliminate factors common to both the numerator and the denominator. A rational expression is simplified when 1 is the only common polynomial factor of both the numerator and the denominator.

EXAMPLE 1 **(Simplify Rational Expressions)**

Simplify each of the following rational expressions:

(a) $\dfrac{x^2-2x-15}{3x-15}$ (b) $\dfrac{7+2x-3x^2}{2x^2-11x-21}$

SOLUTION

(a) $\dfrac{x^2-2x-15}{3x-15}=\dfrac{(x+3)\cancel{(x-5)}}{3\cancel{(x-5)}}=\dfrac{x+3}{3}, \qquad x \neq 5$

(b)

$$\frac{7+20x-3x^2}{2x^2-11x-21}=\frac{(7-x)(1+3x)}{(x-7)(2x+3)}=$$

Factor

$$=\frac{-(x-7)(1+3x)}{(x-7)(2x+3)}=\frac{-\cancel{(x-7)}(1+3x)}{\cancel{(x-7)}(2x+3)}=\frac{-(1+3x)}{2x+3}=-\frac{3x+1}{2x+3}$$

Use (7-x) = - (x – 7).

Caution. Rational expressions can be simplified by eliminating nonzero *factors* common to the numerator and the denominator, but not terms or factors of terms.

Arithmetic operations are defined on rational expressions just as they are not rational numbers.

Arithmetic Operations Defined on Rational Expressions

For all rational expressions *P/Q* and *R/S* where $Q \neq 0$ and $S \neq 0$

1. Addition

$$\frac{P}{Q}+\frac{R}{Q}=\frac{P+R}{Q}$$

2. Subtraction

$$\frac{P}{Q}-\frac{R}{Q}=\frac{P-R}{Q}$$

3. Multiplication

$$\frac{P}{Q}\cdot\frac{R}{S}=\frac{PR}{QS}$$

4. Division

$$\frac{P}{Q}:\frac{R}{S}=\frac{P}{Q}\cdot\frac{S}{R}=\frac{PS}{QR},\quad R\neq 0$$

Factoring often plays a key role in the multiplication and division of rational expressions. Often, the resulting rational expression can be reduced using the equivalent expressions property of rational expressions.

EXAMPLE 2 **(Multiply and Divide Rational Expressions)**

Perform the indicated operation and then reduce if possible:

(a) $\dfrac{x^2+2x}{4x^2-6x}\cdot\dfrac{x^2-16}{x^2+3x-28}$;

(b) $\dfrac{x^2+6x+9}{x^3+27}\div\dfrac{x^2+7x+12}{x^3-3x^2+9x}$.

SOLUTION

(a)

$$\frac{x^2+7x}{4x^2-6x}\cdot\frac{x^2-16}{x^2+3x-28}=\frac{x(x+7)}{2x(2x-3)}\cdot\frac{(x+4)(x-4)}{(x-4)(x+7)}= \qquad \textit{Factor.}$$

$$=\frac{\cancel{x}\,\cancel{(x+7)}\,(x+4)\,\cancel{(x-4)}}{2\cancel{x}\,(2x-3)\,\cancel{(x-4)}\,\cancel{(x+7)}}= \qquad \textit{Simplify.}$$

$$=\frac{x+4}{2(2x-3)} \qquad \textit{The answer.}$$

(b)

$$\frac{x^2+6x+9}{x^3+27}:\frac{x^2+7x+12}{x^3-3x^2+9x}=$$

$=\dfrac{(x+3)^2}{(x+3)(x^2-3x+9)}:\dfrac{(x+4)(x+3)}{x(x^2-3x+9)}=$	*Factor.*
$=\dfrac{(x+3)^2}{(x+3)(x^2-3x+9)}\cdot\dfrac{x(x^2-3x+9)}{(x+4)(x+3)}=$	*Multiply by the reciprocal.*
$=\dfrac{\cancel{(x+3)^2}\,x\,\cancel{(x^2-3x+9)}}{\cancel{(x+3)}\,\cancel{(x^2-3x+9)}\,(x+4)\,\cancel{(x+3)}}=$	*Simplify.*
$=\dfrac{x}{x+4}$	*The answer.*

Addition of rational expressions with a **common denominator** is accomplished by writing the sum of the numerators over the common denominator. For example, $\dfrac{5x}{18}+\dfrac{x}{18}=\dfrac{5x+x}{18}=\dfrac{6x}{18}=\dfrac{x}{3}$.

3.5. LEAST COMMON DENOMINATOR (LCD)

If the rational expressions do not have a common denominator, then they can be written as equivalent rational expressions that have a common denominator by multiplying numerator and denominator of each of the rational expressions by the required polynomials. The following procedure can be used to determine the least common denominator (LCD) of rational expressions. It is similar to the process used to find the LCD of fractions.

Determining the LCD of two or More Rational Expressions

1. Factor each denominator completely and express repeated factors using exponential notation.

2. Identify the largest power of each factor in any single factorization. The LCD is the product of each factor raised to its largest power.

For example, $\dfrac{1}{x+3}$ *and* $\dfrac{5}{2x-1}$ have a LCD of $(x+3)(2x-1)$. The rational expressions $\dfrac{5x}{(x+5)(x-7)^3}$ *and* $\dfrac{7}{x(x+5)^2(x-7)}$ have a LCD of $x(x+5)^2(x-7)^3$.

Example 3 illustrates the process of adding and subtracting rational expressions by writing each rational expression as an equivalent ration expression having the LCD as its denominator.

EXAMPLE 3 **(Add and Subtract Rational Expressions)**

Perform the indicated operation and then simplify if possible:

(a) $\dfrac{5x}{48}+\dfrac{x}{15}$ (b) $\dfrac{x}{x^2-4}-\dfrac{2x-1}{x^2-3x-10}$

SOLUTION

(a) Determine the prime factorization of the denominators: $48 = 2^4 \cdot 3$ and $15 = 3 \cdot 5$. The LCD is the product of each of the prime factors raised to its largest power. Thus the LCD is $2^4 \cdot 3 \cdot 5 = 240$. Write each rational expression as an equivalent rational expression with a denominator of 240.

$$\frac{5x}{48} + \frac{x}{15} = \frac{5x \cdot 5}{48 \cdot 5} + \frac{x \cdot 16}{15 \cdot 16} = \left\langle \text{Use the equivalent expressions property } \frac{P}{Q} = \frac{PR}{QR} \right\rangle =$$

$$= \frac{25x}{240} + \frac{16x}{240} = \frac{41x}{240}.$$

(b) Factor each denominator to determine the LCD of the rational expressions:

$$x^2 - 4 = (x+2)(x-2) \text{ and } x^2 - 3x - 10 = (x+2)(x-5).$$

The LCD is $(x+2)(x-2)(x-5)$. Therefore, forming equivalent rational expressions that have the LCD, we have

$$\frac{x}{x^2-4} - \frac{2x-1}{x^2-3x-10} = \frac{x(x-5)}{(x+2)(x-2)(x-5)} - \frac{(2x-1)(x-2)}{(x+2)(x-5)(x-2)} =$$

$$= \frac{x^2 - 5x - (2x^2 - 5x + 2)}{(x+2)(x-2)(x-5)} = \quad \textit{Subtract}$$

$$= \frac{x^2 - 5x - 2x^2 + 5x - 2}{(x+2)(x-2)(x-5)} = \quad \textit{Simplify}$$

$$= \frac{-x^2 - 2}{(x+2)(x-2)(x-5)} = -\frac{x^2 + 2}{(x+2)(x-2)(x-5)} \quad \textit{The answer}$$

3.6. COMPLEX FRACTIONS

A **complex fraction** is a fraction whose numerator or denominator contains one or more fractions. Complex fractions can be simplified by using one of the following two methods.

Methods for Simplifying Complex Fractions

Method 1: Multiply by the LCD:

1. Determine the LCD of all the fractions in the complex fraction.
2. Multiply both the numerator and the denominator of the complex fraction by the LCD.
3. If possible, simplify the resulting rational expression.

Method 2: Simplify and multiply by the reciprocal of the denominator

1. Simplify the numerator to a single fraction and the denominator to a single fraction.
2. Multiply the numerator by the reciprocal of the denominator.

3. If possible, simplify the resulting rational expression.

***EXAMPLE 4* (Simplify Complex Fractions)**

Simplify each complex fraction:

(a) $\dfrac{3-\dfrac{2}{a}}{1+\dfrac{4}{a}}$ (b) $\dfrac{\dfrac{2}{x-2}+\dfrac{1}{x}}{\dfrac{3x}{x-5}-\dfrac{2}{x-5}}$

SOLUTION

(a) The LCD of all the fractions in the complex fraction is a. Therefore this fraction can be simplified by multiplying both the numerator and the denominator of the complex fraction by a.

$$\frac{3-\frac{2}{a}}{1+\frac{4}{a}}=\frac{\left(3-\frac{2}{a}\right)a}{\left(1+\frac{4}{a}\right)a}=\frac{3a-\left(\frac{2}{a}\right)a}{a+\left(\frac{4}{a}\right)a}=\langle\text{Distribute and simplify}\rangle=\frac{3a-2}{a+4}.$$

(b) This complex fraction is best simplified by the method of first simplifying the numerator to a single fraction and then simplifying the denominator to a single fraction.

$\dfrac{\dfrac{2}{x-2}+\dfrac{1}{x}}{\dfrac{3x}{x-5}-\dfrac{2}{x-5}}=\dfrac{\dfrac{2\cdot x}{(x-2)\cdot x}+\dfrac{1\cdot(x-2)}{x\cdot(x-2)}}{\dfrac{3x-2}{x-5}}=$	*Simplify numerator and denominator*
$=\dfrac{\dfrac{2x+(x-2)}{x(x-2)}}{\dfrac{3x-2}{x-5}}=\dfrac{\dfrac{3x-2}{x(x-2)}}{\dfrac{3x-2}{x-5}}=$	
$=\dfrac{\cancel{3x-2}}{x(x-2)}\cdot\dfrac{x-5}{\cancel{3x-2}}=$	*Multiply by the reciprocal of the denominator and simplify*
$=\dfrac{x-5}{x(x-2)}$	*The answer*

***EXAMPLE 5* (Simplify Complex Fractions with Negative Exponents)**

Simplify the complex fraction $\dfrac{c^{-1}}{a^{-1}+b^{-1}}$

SOLUTION

The fraction written without negative exponents becomes $\dfrac{\frac{1}{c}}{\frac{1}{a}+\frac{1}{b}}$.

Using the method of multiplying the numerator and the denominator by abc, which is the LCD of the denominators, yields the following simplification:

$$\frac{\frac{1}{c}}{\frac{1}{a}+\frac{1}{b}}=\frac{\frac{1}{c}\cdot abc}{\left(\frac{1}{a}+\frac{1}{b}\right)abc}=\frac{\frac{1}{c}\cdot abc}{\frac{1}{a}\cdot abc+\frac{1}{b}\cdot abc}=\frac{ab}{bc+ac}$$

Remark It is a mistake to write $\dfrac{c^{-1}}{a^{-1}+b^{-1}}$ as $\dfrac{a+b}{c}$ because a^{-1} and b^{-1} are *terms* and cannot be treated as factors.

Exercise Set 3.2

In exercises 1 to 11, simplify each rational expression.

1. $\dfrac{2x}{2x-4}$

2. $\dfrac{2x-6}{3x-9}$

3. $\dfrac{x-6}{6-x}$

4. $\dfrac{2x+1}{-2x-1}$

5. $\dfrac{x^2-x-20}{3x-15}$

6. $\dfrac{2x^2-5x-12}{2x^2+5x+3}$

7. $\dfrac{x^3-9x}{x^3+x^2-6x}$

8. $\dfrac{x^3+125}{2x^3-50x}$

9. $\dfrac{a^3+8}{a^2-4}$

10. $\dfrac{y^3-27}{-y^2-11y-24}$

11. $\dfrac{x^2+3x-40}{-x^2+3x+10}$

In Exercises 12 to 30, perform the indicated operation(s). State your results in the simplest form.

12. $\left(\dfrac{-4a}{3b^2}\right)\left(\dfrac{6b}{a^4}\right)$

13. $\left(\dfrac{12x^2y}{5z^4}\right)\left(-\dfrac{25x^2z^3}{15y^2}\right)$

14. $\dfrac{x^2+x}{2x+3}\cdot\dfrac{3x^2+19x+28}{x^2+5x+4}$

15. $\dfrac{x^2-16}{x^2+7x+12}\cdot\dfrac{x^2-4x-21}{x^2-4x}$

16. $\dfrac{3x-15}{2x^2-50}\cdot\dfrac{2x^2+16x+30}{6x+9}$

22. $\dfrac{2x}{3x+1}+\dfrac{5x}{x-7}$

23. $\dfrac{5y-7}{y+4}-\dfrac{2y-3}{y+4}$

24. $\dfrac{6x-5}{x-3}-\dfrac{3x-8}{x-3}$

25. $\dfrac{1}{x}+\dfrac{2}{3x-1}\cdot\dfrac{3x^2+11x-4}{x-5}$

26. $\dfrac{2}{y}-\dfrac{3}{y+1}\cdot\dfrac{y^2-1}{y+4}$

27. $\dfrac{q+1}{q-3}-\dfrac{2q}{q-3}\div\dfrac{q+5}{q-3}$

17. $\dfrac{y^3-8}{y^2+y-6}\cdot\dfrac{y^2+3y}{y^3+2y^2+4y}$

18. $\dfrac{12y^2+28y+15}{6y^2+35y+25}\cdot\dfrac{2y^2-y-3}{3y^2+11y-20}$

19. $\dfrac{p+5}{r}+\dfrac{2p-7}{r}$

20. $\dfrac{2s+5t}{4t}+\dfrac{-2s+3t}{4t}$

21. $\dfrac{x}{x-9}-\dfrac{3x-1}{x^2+7x+12}$

28. $\dfrac{p}{p+5}+\dfrac{p}{p-4}\div\dfrac{p+2}{p^2-p-12}$

29. $\left(1+\dfrac{2}{x}\right)\left(3-\dfrac{1}{x}\right)$

30. $\left(4-\dfrac{1}{z}\right)\left(4+\dfrac{2}{z}\right)$

In Exercises 31 to 42, simplify each complex fraction.

31. $\dfrac{4+\dfrac{1}{x}}{1-\dfrac{1}{x}}$

32. $\dfrac{3-\dfrac{2}{a}}{5+\dfrac{3}{a}}$

33. $\dfrac{\dfrac{x}{y}-2}{y-x}$

34. $\dfrac{3+\dfrac{2}{x-3}}{4+\dfrac{1}{2+\dfrac{1}{x}}}$

35. $\dfrac{5-\dfrac{1}{x+2}}{1+\dfrac{3}{1+\dfrac{3}{x}}}$

36. $\dfrac{\dfrac{1}{(x+h)^2}-1}{h}$

37. $\dfrac{1+\dfrac{1}{b-2}}{1-\dfrac{1}{b+3}}$

38. $\dfrac{\dfrac{x+h+1}{x+h}-\dfrac{x}{x+1}}{h}$

39. $\dfrac{\dfrac{1}{x}-\dfrac{x-4}{x+1}}{\dfrac{x}{x+1}}$

40. $\dfrac{\dfrac{2}{y}-\dfrac{3y-2}{y-1}}{\dfrac{y}{y-1}}$

41. $\dfrac{\dfrac{1}{x+3}-\dfrac{2}{x-1}}{\dfrac{x}{x-1}+\dfrac{3}{x+3}}$

42. $\dfrac{\dfrac{x+2}{x^2-1}+\dfrac{1}{x+1}}{\dfrac{x}{2x^2-x-1}+\dfrac{1}{x-1}}$

Section4. Rational Equations

4.1. LINEAR EQUATIONS

An equation is a statement about the equality of two expressions. If neither expression contains a variable, it is easy to determine whether the equation is a true or a false statement. For example, the equation $2+3=5$ is a true statement. If the expressions contain a variable, the equation may be a true statement for some values of the variable and a false statement for other values of the variable. For example, $3x+2=14$ is a true statement when $x=4$, but it is false for any number except 4.

A number is said to **satisfy** an equation if substituting the number for the variable produces an equation that is a true statement. To **solve** an equation means to find all values of the variable that satisfy the equation. The values that make the equation true are called **solutions** or **roots** of the equation. For example, 5 is a solution or root of $2x-10=0$ because $2(5)-10=0$ is a true statement.

The **solution set** of an equation is the set of all solutions of the equation. The solution set of $x^2-5x+6=0$ is $\{2,3\}$ because 2 and 3 are the only numbers that satisfy the equation.

Equivalent equations have the same solution set. The process of solving an equation is generally accomplished by producing *simpler* but equivalent equations until the solutions are easy to observe. To produce these simpler equivalent equations, we often apply the following properties.

Addition and Multiplication Properties of Equality

For real numbers a, b, and c, $a=b$ and $a+c=b+c$ are equivalent equations.
If $c\neq 0$, then $a=b$ and $ac=bc$ are equivalent equations.

Essentially, these properties state that an equivalent equation is produced by either adding the same expression to each side of an equation or multiplying each side of an equation by the same nonzero expression. For example,

$x-2=7$ and $x-2+2=7+2$ are equivalent equations;

$3x=12$ and $\frac{1}{3}(3x)=\frac{1}{3}(12)$ are equivalent equations.

Definition of a Linear Equation. A **linear equation** in one variable is an equation that can be written in the form $ax+b=0$ where a and b are real numbers, with $a\neq 0$.

The addition and multiplication properties of equality can be used to solve a linear equation.

EXAMPLE 1 **(Solve a Linear Equation)** Solve the liner equation $\frac{3}{4}x-6=0$.

SOLUTION

To isolate the x term on the left side of the equation, add 6 to each side of the equation.

$$\frac{3}{4}x-6=0 \quad\rightarrow\quad \frac{3}{4}x-6+6=0+6 \quad\rightarrow\quad \frac{3}{4}x=6$$

To get the variable x alone on the left side of the equation, multiply each side by $4/3$ (the reciprocal of $3/4$): $\left(\frac{4}{3}\right)\left(\frac{3}{4}x\right)=\left(\frac{4}{3}\right)(6) \quad\rightarrow\quad x=8$.

Check by substituting 8 for x in the original equation.

$$\frac{3}{4}x-6=0\Rightarrow \frac{3}{4}(8)-6=0\Rightarrow 0=0 \qquad \Big|\ \textit{True}$$

The proposed solution, 8, satisfies the original equation. The solution set of $\frac{3}{4}x-6=0$ is $\{8\}$.

If an equation involves fractions, it is convenient to multiply each side of the equation by the LCD of all the denominators to produce an equivalent equation that does not contain fractions.

EXAMPLE 2 **(Solve by Clearing Fractions)**

Solve the equation $\frac{2}{3}x+10-\frac{x}{5}=\frac{36}{5}$.

SOLUTION Multiply each side of the equation by 15, which is the LCD of all the denominators.

$$15\left(\frac{2}{3}x+10-\frac{x}{5}\right)=15\left(\frac{36}{5}\right)\Rightarrow 10x+150-3x=108.$$

Combine like terms on each side of the equation.

$$7x+150=108$$

To isolate the term that involves the variable x, use the addition property of equality to add -150 to each side of the equation. Subtracting 150 is equivalent to adding -150.

$$7x+150-150=108-150$$
$$7x=-42$$

To get the variable x *alone* on the left side of the equation, use the multiplication property of equality to multiply each side by 1/7.

$$\frac{7x}{7}=\frac{-42}{7}$$

$x=-6$. Check as before.

EXAMPLE 3 **(Solve an Equation by Applying Properties)**

Solve the equation $(x+2)(5x+1)=5x(x+1)$.

SOLUTION

$(x+2)(5x+1)=5x(x+1)$

$5x^2+11x+2=5x^2+5x$	*Subtract* $5x^2$ *from each side.*
$11x+2=5x$	*Subtract* $5x$ *from each side.*
$6x+2=0$	*Subtract* 2 *from each side*
$6x=-2$	*Divide each side of the equation by* 6.
$x=-\frac{1}{3}$	

Check as before.

An equation that has no solution is called a **contradiction.** The equation $x=x+1$ is a contradiction. No number is equal to itself increased by 1.

An **identity** is an equation that is true for *every* real number for which all terms of the equation are defined. Example of identities include the equations $x+x=2x$, and

$$(x+3)^2=x^2+6x+9.$$

An equation that is true for some values of the variable but not true for other values of the variable is called a **conditional equation**. For example, $x+2=8$ is a conditional equation because it is true for $x=6$ and false for any number not equal to 6.

The multiplication property of equality states that you can multiply each side of an equation by the same *nonzero* number. If you multiply each side of an equation by an algebraic expression that involves a variable, you must restrict the variable so that the expression is not equal to zero.

EXAMPLE 4 **(Solve Equations That Have restrictions)**

Solve the following equations:

(a) $\dfrac{x}{x-3}=\dfrac{9}{x-3}-5$ (b) $1+\dfrac{x}{x-5}=\dfrac{5}{x-5}$

SOLUTION

(a) First, note that the denominator $x-3$ would equal zero if $x=3$. To produce a simpler equivalent equation, multiply each side by $x-3$, with the restriction that $x\neq 3$

$$(x-3)\left(\frac{x}{x-3}\right)=(x-3)\left(\frac{9}{x-3}-5\right)$$

$$x=(x-3)\left(\frac{9}{x-3}\right)-(x-3)5$$

$$x=9-5x+15$$

Adding $5x$ to each side of the equation produces

$$6x=24$$

$$x=4$$

Since $4\neq 3$, we check by substituting 4 for x in the original equation to show that 4 is indeed a solution. The solution set of our original equation is $\{4\}$.

(b) To produce a simpler equivalent equation, multiply each side of the equation by $x-5$, with the restriction that $x\neq 5$.

$$(x-5)\left(1+\frac{x}{x-5}\right)=(x-5)\left(\frac{5}{x-5}\right)$$

$$(x-5)1+(x-5)\left(\frac{x}{x-5}\right)=5$$

$$x-5+x=5$$

$$2x-5=5$$

$$2x=10$$

$$x=5$$

Although we have obtained 5 as a proposed solution, 5 is *not* a solution of the original equation since it contradicts our restriction $x\neq 5$. Substitution of 5 for x in the original equation results in denominators of 0. In this case, the solution set of the original equation is the empty set.

Exercise Set 4.1

In Exercises 1 to 72, solve the linear equation

1. $\frac{7}{9}n+\frac{11}{12}=\frac{13}{18}$

2. $y-\frac{5}{7}y=\frac{2}{9}$

3. $s-\frac{8}{15}s=\frac{1}{3}$;

4. $2\frac{1}{3}x+1\frac{1}{2}=1\frac{2}{3}x+2\frac{1}{3}$

5. $\frac{1}{2}-1\frac{3}{5}y=4\frac{1}{2}-3y$

6. $\frac{5}{8}s-\frac{3}{4}=2s-2\frac{2}{5}$

7. $\frac{5}{8}n-\frac{7}{12}n+\frac{5}{6}n=\frac{1}{4}$

8. $\frac{2}{3}z+\frac{5}{6}z-\frac{7}{9}z=\frac{1}{2}$

9. $\frac{2}{5}f+\frac{3}{10}f-\frac{2}{15}f=\frac{1}{6}$

10. $\left(\frac{8}{15}+\frac{2}{9}y\right):\frac{3}{5}=2$;

11. $\left(\frac{3}{14}+\frac{5}{21}d\right):\frac{3}{7}=3\frac{1}{4}$

12. $\left(\frac{7}{18}+\frac{5}{24}z\right):3\frac{2}{3}=\frac{1}{3}$

13. $(3x-1)\cdot 0{,}2=\frac{1}{5}(x+5)$

14. $\frac{2x+9}{6}=\frac{6-9x}{4}$

15. $z-\frac{11}{18}=-\frac{5}{6}$

16. $4x-7=3{,}4$

17. $5x-8=4{,}5$

18. $4{,}8-0{,}3x=0$

19. $-0{,}4y+5{,}2=0$

20. $x-(-2{,}7)=3{,}8$

21. $-3.2+x=5.2$

22. $z\div 5.4=10.2\div 1.8$

23. $0{,}8a+1{,}4=0{,}4a-2{,}6$

24. $-0{,}2x\cdot(-0{,}7)=0{,}84$

25. $7{,}3-(y-3{,}7)=15$

26. $-3{,}8+(x-4{,}2)=3{,}6$

27. $3{,}6+2x=5x+1{,}2$

28. $0{,}7s-1{,}82=0{,}8s+3{,}46$

29. $4{,}72-2{,}5d=2d+2{,}92$

30. $4(1-0{,}5b)=-2(3+2b)$

31. $5(t+1{,}2)=12{,}5t$

32. $4(0{,}2s-7)-5(0{,}3s+6)=5$

33. $2{,}3\cdot(2x+0{,}1)=4{,}6x+0{,}1$

34. $3(0{,}4a+7)-4(0{,}8a-3)=2$

35. $6\frac{1}{12}-k=2\frac{5}{12}$

36. $a+\frac{2}{9}=\frac{5}{6}$

37. $\frac{7}{8}y=1\frac{1}{4}$

38. $\frac{2}{15}s+\frac{3}{4}=\frac{5}{6}$

39. $2\frac{1}{15}-\frac{3}{4}m=\frac{59}{60}$

40. $x+6=10$

41. $a+15=4$

42. $10+y=19$

43. $24+z=17$

44. $x-6=13$

45. $y-9=-16$

46. $7-t=5$

47. $9-s=-10$

48. $4\cdot x=20$

49. $-9\cdot x=27$

50. $x\cdot 7=35$

51. $x\cdot(-12)=24$

52. $15\div x=3$

53. $24\div y=-4$

54. $a\div 3=-25$

55. $b\div 2=8$

56. $5x+3=2$

57. $2x+7=19$

58. $3+5y=-12$

59. $120-6x=15$

60. $3t-7=-10$

61. $19-6s=7$

62. $5^2+2x=3$

63. $6+3(x+9)=48$

64. $9x+14=6-3x$

65. $4x+12=3x+8$

66. $3x-17=8x+18$

67. $14+5x=4x+3x$

68. $5(2x-1)=8x+1$

69. $3a+5=8a-15$

70. $-3{,}2+x=5{,}2$

71. $6{,}7+y=-4{,}3$

72. $5{,}7-a=8{,}9$

4.2. QUADRATIC EQUATIONS

4.2.1. The Quadratic Formula

Completing the square on the standard quadratic form $ax^2+bx+c=0,(a\neq 0)$, produces a formula for x in terms of the coefficients $a,b,$ and c. The formula is known as the **quadratic formula**, and it is another method for solving quadratic equations.

The Quadratic Formula

If $ax^2+bx+c=0$, $a\neq 0$, then $x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$.

Proof : We assume a is a positive real number .If a were a negative real number, then we could multiply each side of the equation by -1 to make it positive.

$ax^2+bx+c=0,(a\neq 0)$	*Given*
$ax^2+bx=-c$	*Isolate the variable terms*
$x^2+\dfrac{b}{a}x=-\dfrac{c}{a}$	*Multiply each term on each side of the equation by* $\dfrac{1}{a}$
$x^2+\dfrac{b}{a}x+\left(\dfrac{b}{2a}\right)^2=\left(\dfrac{b}{2a}\right)^2-\dfrac{c}{a}$	*Complete the square*
$\left(x+\dfrac{b}{2a}\right)^2=\left(\dfrac{b}{2a}\right)^2-\dfrac{c}{a}$	*Factor the left side*
$\left(x+\dfrac{b}{2a}\right)^2=\dfrac{b^2}{4a^2}-\dfrac{4a}{4a}*\dfrac{c}{a}$	*Simplify the right side*
$x+\dfrac{b}{2a}=\pm\sqrt{\dfrac{b^2-4ac}{4a^2}}$	*Apply the square root theorem*
$x+\dfrac{b}{2a}=\dfrac{\pm\sqrt{b^2-4ac}}{2a}$	*Since* $a>0$, $\sqrt{4a^2}=2a$.
$x=-\dfrac{b}{2a}\pm\dfrac{\sqrt{b^2-4ac}}{2a}$	*Add* $-\dfrac{b}{2a}$ *to each side*

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

To solve a quadratic equation using the quadratic formula, first write the quadratic equation in the standard quadratic form $ax^2+bx+c=0$.

***EXAMPLE 5* (Solve by Using the quadratic Formula)**

Solve the quadratic equation $3x^2 = -2x+5$.

SOLUTION

First write the equation in a standard quadratic form and then identify the values of a, b, and c. The equation $3x^2+2x-5=0$ has coefficients $a=3,\ b=2$ and $c=-5$.

$$x_{1,2} = \frac{-(2) \pm \sqrt{2^2 - 4(3)(-5)}}{2(3)}$$ *Substitute in the quadratic formula*

$$x_{1,2} = \frac{-2 \pm \sqrt{64}}{6} = \frac{-2 \pm 8}{6} = 1 \quad \text{or} \quad -\frac{5}{3}.$$

The solution set of $3x^2 = -2x+5$ is $\left\{-\frac{5}{3}, 1\right\}$.

The equation $3x^2 = -2x+5$ could have been solved by factoring and using the zero product property. As a general rule, you should first try to solve quadratic equations by factoring. If the factoring process proves difficult, then solve by using the quadratic formula.

***EXAMPLE 6* (Solve by Using the Quadratic Formula)**

Solve the quadratic equation $4x^2 - 8x + 1 = 0$.

SOLUTION

In this example the coefficients are $a=4,\ b=-8,$ and $c=1$.

$$x_{1,2} = \frac{-(-8) \pm \sqrt{(-8)^2 - 4(4)(1)}}{2(4)} = \frac{8 \pm \sqrt{48}}{8} = \frac{8 \pm 4\sqrt{3}}{8} = \frac{2 \pm \sqrt{3}}{2}.$$

The solution set of $4x^2 - 8x + 1 = 0$ is $\left\{\frac{2-\sqrt{3}}{2}, \frac{2+\sqrt{3}}{2}\right\}$.

4.2.2. The Discriminant

In the quadratic formula $x_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$, the expression $b^2 - 4ac$ is called the **discriminant** of the quadratic formula. The quadratic formula involves the square root of the discriminant. If $b^2 - 4ac \geq 0$, then $\sqrt{b^2 - 4ac}$ is a real number; if $b^2 - 4ac < 0$, then $\sqrt{b^2 - 4ac}$ is a complex number.

Thus the sign of the discriminant determines whether the roots of a quadratic equation are real numbers or complex numbers.

The Discriminant and Roots of a Quadratic Equation

The quadratic equation $ax^2 + bx + c = 0$, with real coefficients and $a \neq 0$, has discriminant $b^2 - 4ac$.

If $b^2 - 4ac > 0$, then the quadratic equation has *two distinct real roots.*

If $b^2 - 4ac = 0$ then the quadratic equation has *a real root* that is a double root.

If $b^2 - 4ac < 0$, then the quadratic equation has *two distinct nonreal complex roots.*

By examining the discriminant, it is possible to determine whether the roots of a quadratic equation are real numbers or complex numbers without actually finding the roots.

EXAMPLE 7 **(Use the Discriminant to Classify roots)**

Classify the roots of each quadratic equation as real numbers or complex numbers.

(a) $2x^2-5x+1=0$ (b) $3x^2+6x+7=0$ (c) $x^2+6x+9=0$

SOLUTION

(a) $2x^2-5x+1=0$ has coefficients $a=2, b=-5$ and $c=1$.

$b^2-4ac=(-5)^2-4(2)(1)=25-8=17$

Because the discriminant 17 is *positive,* the equation $2x^2-5x+1=0$ has *two distinct real roots*.

(b) $3x^2+6x+7=0$ has coefficients $a=3, b=6$ and $c=7$.

$b^2-4ac=6^2-4(3)(7)=36-84=-48$

Because the discriminant –48 is *negative*, $3x^2+6x+7=0$ has *two distinct complex roots*.

(c) $x^2+6x+9=0$ has coefficients $a=1, b=6$ and $c=9$.

$b^2-4ac=6^2-4(1)(9)=36-36=0$

Because the discriminant is 0, the equation $x^2+6x+9=0$ has a real root. The root is a double root.

4.2.3. The Sum and Product of the Roots Theorem.

The roots of $x^2+2x-15=0$ are -5 and 3. Notice that the sum of the roots equals the opposite of the coefficient of the x term and the product of the roots equals the constant term. This example is a special case of the following theorem.

Theorem If $a\neq 0$ and r_1 and r_2 are roots of $ax^2+bx+c=0$, which has the equivalent for $x^2+\frac{b}{a}x+\frac{c}{a}=0$. Then the sum of the roots $r_1+r_2=-\frac{b}{a}$, the product of the roots $r_1r_2=\frac{c}{a}$.

One method of checking the solutions or roots of an equation is to substitute the roots into the original equation. In Example9, the sum and product of the roots theorem is used to check the proposed roots of quadratic equations.

EXAMPLE 8 **(Check by Using the Sum and Product of the Roots Theorem)**

Check the proposed roots of each quadratic equation:

(a) $x^2-2x-15=0$ (Proposed roots:-3, 5);

(b) $x^2-4x-4=0$ (Proposed roots: $2\pm 2\sqrt{2}$).

SOLUTION

(a) The sum of the proposed roots is $(-3)+5=2$. Their product is $(-3)(5)=-15$.

The proposed roots are checked because their sum, 2, equals $-\frac{b}{a}=-\frac{-2}{1}=2$, and their product, -15, equals $\frac{c}{a}=\frac{-15}{1}=-15$.

(b) The sum of proposed roots is $\left(2-2\sqrt{2}\right)+\left(2+2\sqrt{2}\right)=4$.

Their product is $\left(2-2\sqrt{2}\right)\left(2+2\sqrt{2}\right)=4-8=-4$. The proposed roots are checked because their sum, 4, equals $-\frac{b}{a}=-\frac{-4}{1}=4$, and their product, -4, equals $\frac{c}{a}=\frac{-4}{1}=-4$.

Exercise Set 4.2

In Exercises 1 to 58, solve the quadratic equation

1. $25x^2=4$;
2. $x^2-144=0$;
3. $3x^2+26=1$;
4. $14x^2-84=266$;
5. $8x^2=-72$;
6. $36x^2-19=7x^2$;
7. $36x^2+9=0$;
8. $x^2-1=35$;
9. $4x^2=144$;
10. $x^2=8$;
11. $3x^2=363$;
12. $4x^2-121=0$;
13. $x^2-42=-26$;
14. $2x^2-72=0$;
15. $9x^2=7$;
16. $19x^2=50+17x^2$.

In Exercises 1 to 58, solve the quadratic equation

17. $x^2+3x=0$;
18. $-x^2-73x=0$;
19. $\frac{1}{5}x^2-2x=0$;
20. $-18x^2=36x$;
21. $-x^2+14x=0$;
22. $5x^2+17x=0$;
23. $\frac{13}{15}x-17x^2=0$;
24. $-31x^2=7x$;
25. $x^2-18x=0$;
26. $4x-17x^2=0$;
27. $23x^2=x$;
28. $23x^2=56x$.

In Exercises 1 to 58, classify the roots of each quadratic equation as real numbers or complex numbers.

29. $2x^2+5x-7=0$;
30. $x^2-6x+3=0$;
31. $x^2=-14x-33$;
32. $x^2-2x-15=0$;
33. $3x^2+8x-8=0$;
34. $4x^2+13x+7=0$;
35. $5x^2-6x-9=0$;
36. $3x^2+4x-7=0$;
37. $16x^2+21x-22=0$;
38. $5x^2+9x+2=0$;
39. $-6x^2+7x+9=0$;
40. $-6x^2+7x+2=0$;
41. $6x^2-7x+2=0$;
42. $5x^2+8x+3=0$;
43. $x^2+22x+21=0$;
44. $4x-x^2-1=0$;
45. $-4x^2+3x+9=0$;
46. $-2x^2+8x+13=0$.

In Exercises 1 to 58, solve the quadratic equation. Find the sum and product of the roots.

47. $-x^2+12x-61=0$;
48. $x^2+x+8=0$;
49. $4x^2-4x+1=0$;
51. $2x^2+8=-x$;
52. $2x^2+4x+4=0$;
53. $25x^2+20x+4=0$;
55. $3x^2-x+10=0$;
56. $11x^2-5x+1=0$;
57. $15x^2-60x+36=0$;

50. $x^2+4{,}4x+4{,}84=0$; **54.** $121x^2+44x+4=0$; **58.** $169x^2-24x+1=0$.

Supplemental exercises

1. $(y-4)\cdot(y+3)=-12$

2. $3\cdot(29-5x)=x\cdot(x-5)-10x$

3. $(4x-1)^2-(5x+2)^2+(8x-7)\cdot(8x+7)=28\cdot(6-x)$

4. $(2x-7)^2+(3x-5)^2-2\cdot(64-29)=(4x-9)\cdot(9+4x)$

5. $\frac{9y^2-2y}{3}=\frac{y^2+5y}{4}$

6. $\frac{x^2-5}{2}+\frac{x^2-1}{8}=\frac{x^2-3}{4}+\frac{x^2-4}{3}$

7. $(x+6)\cdot(8-x)=48$

8. $(2x+5)^2=16+(x-3)^2$

9. $\frac{74-2y^2}{12}-10=\frac{11-3y^2}{8}$

10. $x^2-\frac{x^2-1}{3}-\frac{2x^2-5}{5}+\frac{x^2+8}{6}=7$

11. $y^2+4\sqrt{2}y-10=0$

12. $5y^2-2y+2\sqrt{3}-15=0$.

13. $4y^2-3y-2\sqrt{3}=0$

4.3 SYSTEMS OF EQUATIONS

A **system of equations** is a set of equations that involve the same variables. A **solution** of a system is an assignment of values for the variables that makes *each* equation in the system true. To **solve** a system means to find all solutions of the system.

Here is an example of a system of two equations in two variables:

$$\begin{cases} x+4y=3 & \quad Equation\,1 \\ 2x+7y=4 & \quad Equation\,2 \end{cases}$$

We can check that $x=-5$ and $y=2$ is a solution of this system.

Equation 1	***Equation 2***
$x+4y=3$	$2x+7y=4$
$-5+4\cdot 2=3$ *true*	$2\cdot(-5)+7\cdot 2=4$ *true*

The solution can also be written as the ordered pair $(-5,2)$.

4.3.1 Substitution Method

In the substitution method we start with one equation in the system and solve for one variable in terms of the other variable. The following box describes the procedure.

1. ***Solve for One Variable***. Choose one equation and solve for one variable in terms of the other variable.

2. ***Substitute***. Substitute the expression you found in Step 1 into the other equation to get an equation in one variable, then solve for that variable.

3. ***Back-Substitute***. Substitute the value you found in Step 2 back into the expression found in Step 1 to solve for the remaining variable.

EXAMPLE 1 **(Substitution Method)**

Find all solutions of the system.

$$\begin{cases} 2x + y = 1 & Equation 1 \\ 3x + 4y = 14 & Equation 2 \end{cases}$$

SOLUTION

We solve for y in the first equation.

$$y = 1 - 2x \qquad \Big| \ \textit{Solve for } y \textit{ in Equation 1}$$

Now we substitute for y in the second equation and solve for x:

$$\begin{array}{c|l} 3x + 4(1-2x) = 14 & \textit{Substitute } y = 1 - 2x \textit{ into Equation 2} \\ 3x + 4 - 8x = 14 & \textit{Expand} \\ -5x + 4 = 14 & \textit{Simplify} \\ -5x = 10 & \textit{Subtract 4} \\ x = -2 & \textit{Solve for } x \end{array}$$

Next we back-substitute $x = -2$ into the equation $y = 1 - 2x$:

$$y = 1 - 2 \cdot (-2) = 5 \qquad \Big| \ \textit{Back-substitute}$$

Thus, $x = -2$ and $y = 5$, so the solution is the ordered pair $(-2, 5)$.

EXAMPLE 2 **(Substitution Method)**

Find all solutions of the system.

$$\begin{cases} x^2 + y^2 = 100 & Equation 1 \\ 3x - y = 10 & Equation 2 \end{cases}$$

SOLUTION

We start by solving for y in the second equation.

$$y = 3x - 10 \qquad \Big| \ \textit{Solve for } y \textit{ in Equation 2}$$

Next we substitute for y in the first equation and solve for x:

$$\begin{array}{c|l} x^2 + (3x-10)^2 = 100 & \textit{Substitute } y = 3x - 10 \textit{ into Equation 1} \\ x^2 + (9x^2 - 60x + 100) = 100 & \textit{Expand} \\ 10x^2 - 60x = 0 & \textit{Simplify} \\ 10x(x-6) = 0 & \textit{Factor} \\ x = 0 \quad or \quad x = 6 & \textit{Solve for } x \end{array}$$

Now we back-substitute these values of x into the equation $y = 3x - 10$.

$$\begin{array}{l|l} \text{For } x = 0\text{: } y = 3 \cdot 0 - 10 = -10 & \textit{Back-substitute} \\ \text{For } x = 6\text{: } y = 3 \cdot 6 - 10 = 8 & \textit{Back-substitute} \end{array}$$

So we have two solutions: $(0, -10)$ and $(6, 8)$.

4.3.2 Elimination Method

To solve a system using the **elimination method**, we try to combine the equations using sums or differences so as to eliminate one of the variables.

1. ***Adjust the Coefficients***. Multiply one or more equations by appropriate numbers so that the coefficient of one variable in one equation is the negative of its coefficient in the other equation.

2. ***Add the Equations***. Add the two equations to eliminate one variable, then solve for the remaining variable.

3. ***Back-Substitute***. Substitute the value you found in Step 2 back into one of the original equations, and solve for the remaining variable.

EXAMPLE 3 **(Elimination Method)**

Find all solutions of the system.

$$\begin{cases} 3x+2y=14 & \textit{Equation}\,1 \\ x-2y=2 & \textit{Equation}\,2 \end{cases}$$

SOLUTION

Since the coefficients of the y-terms are negatives of each other, we can add the equations to eliminate y.

$$\begin{array}{c|l} \begin{cases} 3x+2y=14 \\ x-2y=2 \end{cases} & \textit{System} \\ 4x+0\cdot y=16 & \textit{Add} \\ x=4 & \textit{Solve for } x \end{array}$$

Now we back-substitute $x=4$ into one of the original equations and solve for y. Let's choose the second equation because it looks simpler.

$$\begin{array}{c|l} x-2y=2 & \textit{Equation 2} \\ 4-2y=2 & \textit{Back-substitute } x=4 \textit{ into Equation 2} \\ -2y=-2 & \textit{Subtract 4} \\ y=1 & \textit{Solve for y} \end{array}$$

The solution is $(4,1)$.

EXAMPLE 4 **(Elimination Method)**

Find all solutions of the system.

$$\begin{cases} 3x^2+2y=26 & \textit{Equation}\,1 \\ 5x^2+7y=3 & \textit{Equation}\,2 \end{cases}$$

SOLUTION

We choose to eliminate the x-term, so we multiply the first equation by 5 and the second equation by -3. Then we add the two equations and solve for y.

$$\begin{array}{c|l} \begin{cases} 3x^2+2y=26 \\ 5x^2+7y=3 \end{cases} & \begin{array}{l} 5\times \textit{ Equation 1} \\ -3\times \textit{ Equation 2} \end{array} \\ -11y=121 & \textit{Add} \\ y=-11 & \textit{Solve for } y \end{array}$$

Now we back-substitute $y=-11$ into one of the original equations, say $3x^2+2y=26$ and solve for x:

$3x^2+2\cdot(-11)=26$	*Back-substitute $y=-11$ into Equation 1*
$3x^2=48$	*Add 22*
$x^2=16$	*Divide by 3*
$x=-4$ *or* $x=4$	*Solve for x*

So we have two solutions: $(-4,-11)$ and $(4,-11)$.

A **system of two linear equations in two variables** has the form

$$\begin{cases} a_1x+b_1y=c_1 \\ a_2x+b_2y=c_2 \end{cases}$$

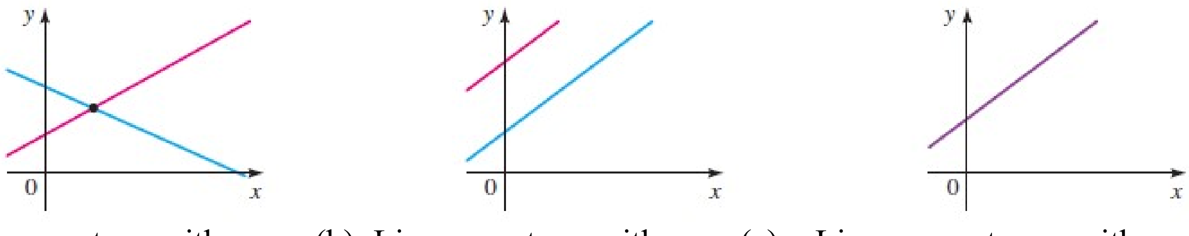
We can use either the substitution method or the elimination method to solve such systems algebraically. But since the elimination method is usually easier for linear systems, we use elimination rather than substitution in our examples.

The graph of a linear system in two variables is a pair of lines, so to solve the system graphically, we must find the intersection point(s) of the lines. Two lines may intersect in a single point, they may be parallel, or they may coincide, as shown in Figure 2. So there are three possible outcomes when solving such a system.

Number of Solutions of a Linear System in Two Variables

For a system of linear equations in two variables, exactly one of the following is true.

1. The system has exactly *one solution* (see Figure 2a).
2. The system has *no solution* (see Figure 2b).
3. The system has *infinitely many solutions* (see Figure 2c).



(a) Linear system with one solution. Lines intersect at a single point.

(b) Linear system with no solution. Lines are parallel—they do not intersect

(c) Linear system with infinitely many solutions. Lines coincide—equations are for the same line.

Figure 2

Definition A system that has no solution is said to be **inconsistent**. A system with infinitely many solutions is called **dependent**.

EXAMPLE 5 **(A Linear System with One Solution)**

Solve the system.

$$\begin{cases} 4x+3y=14 & \qquad Equation\,1 \\ x-3y=1 & \qquad Equation\,2 \end{cases}$$

SOLUTION

Since the coefficients of the y-terms are negatives of each other, we can add the equations to eliminate y.

$$\begin{array}{r|l} \begin{cases} 4x+3y=14 \\ x-3y=1 \end{cases} & \textit{System} \\ 5x+0\cdot y=15 & \textit{Add} \\ x=3 & \textit{Solve for } x \end{array}$$

Now we back-substitute $x=3$ into one of the original equations and solve for y. Let's choose the second equation because it looks simpler.

$$\begin{array}{r|l} x-3y=1 & \textit{Equation 2} \\ 3-3y=1 & \textit{Back-substitute } x=3 \textit{ into Equation 2} \\ -3y=-2 & \textit{Subtract 4} \\ y=\dfrac{2}{3} & \textit{Solve for y} \end{array}$$

The solution is $\left(3,\dfrac{2}{3}\right)$. The lines in the system intersect at the point $\left(3,\dfrac{2}{3}\right)$.

EXAMPLE 6 **(A Linear System with No Solution)**

Solve the system $\begin{cases} 8x-2y=5 & \qquad Equation\ 1 \\ -12x+3y=7 & \qquad Equation\ 2 \end{cases}$.

SOLUTION

This time we try to find a suitable combination of the two equations to eliminate the variable y. Multiplying the first equation by 3 and the second by 2 gives

$$\begin{array}{r|l} \begin{cases} 8x-2y=5 \\ -12x+3y=7 \end{cases} & \begin{array}{l} 3\times \textit{ Equation 1} \\ 2\times \textit{ Equation 2} \end{array} \\ 0=29 & \textit{Add} \end{array}$$

Adding the two equations eliminates both x and y in this case, and we end up with $0=29$, which is obviously false. No matter what values we assign to x and y, we cannot make this statement true, so the system has no solution. The lines in the system are parallel and do not intersect. The system is inconsistent.

EXAMPLE 7 **(A Linear System with Infinitely Many Solutions)**

Solve the system $\begin{cases} 3x-6y=12 & \qquad Equation\ 1 \\ 4x-8y=16 & \qquad Equation\ 2 \end{cases}$.

SOLUTION

We multiply the first equation by 4 and the second by 3 to prepare for subtracting the equations to eliminate x. The new equations are

$$\begin{cases} 12x - 24y = 48 \\ 12x - 24y = 48 \end{cases} \quad \begin{array}{|l} 4\times \textit{ Equation 1} \\ 3\times \textit{ Equation 2} \end{array}$$

We see that the two equations in the original system are simply different ways of expressing the equation of one single line. The coordinates of any point on this line give a solution of the system. Writing the equation in slope-intercept form, we have $y = \frac{1}{2}x - 2$. So if we let t represent any real number, we can write the solution as

$$x = t$$
$$y = \frac{1}{2}t - 2.$$

We can also write the solution in ordered-pair form as $\left(t, \frac{1}{2}t - 2\right)$, where t is any real number. The system has infinitely many solutions.

Exercise Set 4.3

In Exercises 1 to 15, use the substitution method to find all solutions of the system of equations.

1. $\begin{cases} x + 4y = 6, \\ 2x + 7y = 11. \end{cases}$ **2.** $\begin{cases} 2x + 3y = 5, \\ x + 2y = 4. \end{cases}$ **3.** $\begin{cases} x - 2y = 8, \\ 3x + 2y = -1. \end{cases}$

4. $\begin{cases} x_1 - 3x_2 = 9, \\ 2x_1 - 4x_2 = -3. \end{cases}$ **5.** $\begin{cases} x - 2 = 0, \\ 2x - 3y - 1 = 0. \end{cases}$ **6.** $\begin{cases} x - y + 1 = 0, \\ x - y = 0. \end{cases}$

7. $\begin{cases} x - 2y = 3, \\ 2x - y = 1. \end{cases}$ **8.** $\begin{cases} x + 2y = 6, \\ x^2 - xy - y^2 = 11. \end{cases}$ **9.** $\begin{cases} x - 3y = 0, \\ x^2 - 5xy + y = 5. \end{cases}$

10. $\begin{cases} x + 2y = 5, \\ x^2 - 2xy + 3y^2 = 6. \end{cases}$ **11.** $\begin{cases} x + y = 1, \\ x^3 + y^3 = 7. \end{cases}$ **12.** $\begin{cases} 3x - 5y - 4 = 0, \\ x + 2y + 6 = 0. \end{cases}$

13. $\begin{cases} x + y = 3, \\ 2x^2 - 5xy - 3y^2 = -5. \end{cases}$ **14.** $\begin{cases} 2x - y = 3, \\ x - xy + y^2 = 3. \end{cases}$ **15.** $\begin{cases} 3x - 2y = 5, \\ x + 2y = 4. \end{cases}$

In Exercises 16 to 27, find all solutions of the system of equations.

16. $\begin{cases} 3x - 5y = -18, \\ 2x - 3y = -11. \end{cases}$ **17.** $\begin{cases} 3x_1 + 4x_2 = 8, \\ 4x_1 - 5x_2 = 1. \end{cases}$ **18.** $\begin{cases} 5x_1 + 4x_2 = -1, \\ 3x_1 - 6x_2 = 5. \end{cases}$

19. $\begin{cases} 2x_1 + 5x_2 = 9, \\ 5x_1 + 7x_2 = 8. \end{cases}$ **20.** $\begin{cases} 7x_1 + 2x_2 = 0, \\ 2x_1 + x_2 = -3. \end{cases}$ **21.** $\begin{cases} 3x_1 - 8x_2 = 1, \\ 4x_1 + 5x_2 = -2. \end{cases}$

22. $\begin{cases} 5x_1 + 4x_2 = -3, \\ 2x_1 - 4x_2 = 0. \end{cases}$ **23.** $\begin{cases} (x + y)^2 - 4(x + y) = 45, \\ (x - y)^2 - 2(x - y) = 3. \end{cases}$ **24.** $\begin{cases} 3x^2 - 5xy + 2y^2 = 0, \\ x^2 + 4xy - 3y^2 = 1. \end{cases}$

25. $\begin{cases} 2x+3y=5, \\ x+2y=4. \end{cases}$ **26.** $\begin{cases} x-2y=8, \\ 3x+2y=-1. \end{cases}$ **27.** $\begin{cases} x+4y=6, \\ 2x+7y=11. \end{cases}$

APPENDIX

MULTIPLICATION TABLE				
1x1=1	2x1=2	3x1=3	4x1=4	5x1=5
1x2=2	2x2=4	3x2=6	4x2=8	5x2=10
1x3=3	2x3=6	3x3=9	4x3=12	5x3=15
1x4=4	2x4=8	3x4=12	4x4=16	5x4=20
1x5=5	2x5=10	3x5=15	4x5=20	5x5=25
1x6=6	2x6=12	3x6=18	4x6=24	5x6=30
1x7=7	2x7=14	3x7=21	4x7=28	5x7=35
1x8=8	2x8=16	3x8=24	4x8=32	5x8=40
1x9=9	2x9=18	3x9=27	4x9=36	5x9=45
1x10=10	2x10=20	3x10=30	4x10=40	5x10=50
6x1=6	7x1=7	8x1=8	9x1=9	10x1=10
6x2=12	7x2=14	8x2=16	9x2=18	10x2=20
6x3=18	7x3=21	8x3=24	9x3=27	10x3=30
6x4=24	7x4=28	8x4=32	9x4=36	10x4=40
6x5=30	7x5=35	8x5=40	9x5=45	10x5=50
6x6=36	7x6=42	8x6=48	9x6=54	10x6=60
6x7=42	7x7=49	8x7=56	9x7=63	10x7=70
6x8=48	7x8=56	8x8=64	9x8=72	10x8=80
6x9=54	7x9=63	8x9=72	9x9=81	10x9=90
6x10=60	7x10=70	8x10=80	9x10=90	10x10=100

TABLE OF SQUARES				
$2^2 = 4$	$6^2 = 36$	$10^2 = 100$	$14^2 = 196$	$18^2 = 324$
$3^2 = 9$	$7^2 = 49$	$11^2 = 121$	$15^2 = 225$	$19^2 = 361$
$4^2 = 16$	$8^2 = 64$	$12^2 = 144$	$16^2 = 256$	$20^2 = 400$
$5^2 = 25$	$9^2 = 81$	$13^2 = 169$	$17^2 = 289$	$25^2 = 625$

Make sure you are thoroughly familiar with the following concepts before attempting the test.

Properties of Rational Exponents

$$a^p \cdot a^q = a^{p+q} \qquad a^p : a^q = a^{p-q} \qquad \left(a^p\right)^q = a^{p \cdot q}$$

$$\left(a^p \cdot b^q\right)^r = a^{pr} \cdot b^{qr} \qquad \left(\frac{a^p}{b^q}\right)^r = \frac{a^{pr}}{b^{qr}} \qquad \sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{ab}$$

$$\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}} \qquad \sqrt[n]{\sqrt[m]{a}} = \sqrt[nm]{a}$$

Arithmetic Operations Defined on Rational Expressions

1. Addition	$\frac{P}{Q}+\frac{R}{Q}=\frac{P+R}{Q}$
2. Substraction	$\frac{P}{Q}-\frac{R}{Q}=\frac{P-R}{Q}$
3. Multiplication	$\frac{P}{Q}\cdot\frac{R}{S}=\frac{PR}{QS}$
4. Division	$\frac{P}{Q}:\frac{R}{S}=\frac{P}{Q}\cdot\frac{S}{R}=\frac{PS}{QR},\quad R\neq 0$

Factoring Formulas

Difference of two squares	$x^2-y^2=(x-y)(x+y)$
Perfect square trinomials	$x^2+2xy+y^2=(x+y)^2$ $x^2-2xy+y^2=(x-y)^2$
Sum of cubes	$x^3+y^3=(x+y)(x^2-xy+y^2)$
Difference of cubes	$x^3-y^3=(x-y)(x^2+xy+y^2)$
Cube of sum	$x^3+3x^2y+3xy^2+y^3=(x+y)^3$
Cube of difference	$x^3-3x^2y+3xy^2-y^3=(x-y)^3$
Factor Trinomial	$ax^2+bx+c=a(x-x_1)(x-x_2)$ $x_{1,2}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

Cumulative Review Tests for Section 1

Test 1

1. Determine which of the numbers are integers $0, \frac{1}{2}, \sqrt{3}, -\frac{4}{2}, \frac{3}{5}, \sqrt{16}$

2. Find the additive inverse of $1\frac{3}{5}$.

3. Which of numbers $0,1,2,3,4,6,9$ are prime?

4. Calculate $3-(6-5)\cdot 4$

5. Calculate $6\frac{1}{4}-\frac{2}{3}$

6. Calculate $3\frac{1}{2}\div 4\frac{2}{3}$

7. Calculate $\dfrac{2\frac{3}{4}\div 1.1+3\frac{1}{3}}{2.5-0.4\cdot 3\frac{1}{3}}$

Test 2

1. Find the value of expression $|4|-|-3|$

2. If $A=\{1;2;3\}$, $B=\{3;4;5\}, C=\{1;3\}$, find: a) $A\cup B$; b) $A\cap C$.

3. Factorize given number: 72.

4. Graph each inequality and write the inequality using interval notation $x\geq -2$ *and* $x<1$

5. Graph each interval and write each interval as inequality $(-4,2]\cup(3,5)$

6. Compare two numbers: 3,14 and 3,2

Cumulative Review Test for Section 2

Test 1

Evaluate each expression

1. $(-2)^3$ **2.** $\dfrac{3^2\cdot 3^5}{3^6}$ **3.** $\left(\dfrac{2^{-3}\cdot 3^2}{2^{-4}\cdot 3}\right)^{-1}$

Simplify each exponential expression so that all exponents are positive.

4. $-4x^5y^2\cdot 3xy^4$ **5.** $\frac{2}{3}x^2y^8\cdot\left(-\frac{1}{2}xy^3\right)^4$

6. $\left(-3a^3b^2\right)^4\cdot\dfrac{5}{\left(3a^2b^3\right)^3}$

7. Evaluate expression $\dfrac{6^3\cdot 8^3\cdot 18}{27^2\cdot 8^2\cdot 9^2\cdot 6^3}$

Cumulative Review Test for Section 3

Test 1

Perform the indicated operation

1. $\left(3x^2-4x\right)+\left(5x+8\right)$

2. $\left(3x^2-4x\right)-\left(5x^2+8x-3\right)$

3. $\left(2x-3\right)\left(x-5\right)$

4. $3\left(x-5\right)-2\left(3x+1\right)$

5. $\left(x-5\right)^2$

Factor each polynomial:

6. x^2-9

7. x^2-5x+4

Simplify each rational expression

8. $\dfrac{4x}{2x+4}$ **9.** $\dfrac{x^2-49}{x+7}$

Cumulative Review Test for Section 4

Test 1

1. Solve the liner equation $\dfrac{2x+9}{6}=\dfrac{6-9x}{4}$

2. Solve the liner equation $\left(x+2\right)\left(5x+1\right)=5x\left(x+1\right)$.

3. Solve the quadratic equation $x^2-2x-15=0$

4. Solve the quadratic equation $(x+6)\cdot(8-x)=48$

5. Solve the quadratic equation $\dfrac{2x}{3x+1}=\dfrac{5x}{x-7}$

6. Find all solutions of the system $\begin{cases} x-2y=3, \\ 2x-y=1. \end{cases}$

7. Find all solutions of the system $\begin{cases} 3x-5y=-18, \\ 2x-3y=-11. \end{cases}$

8. Find all solutions of the system $\begin{cases} x+2y=6, \\ x^2-xy-y^2=11. \end{cases}$

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